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ON THE MOTIONS OF AN OSCILLATING SYSTEM UNDER
THE INFLUENCE OF FLIP-FLOP CONTROLS

By I. Flügge-Lotz and K. Klotter

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ON THE MOTIONS OF AN OSCILLATING SYSTEM UNDER
THE INFLUENCE OF FLIP-FLOP CONTROLS*

By I. Flügge-Lotz and K. Klotter

PART I.-- MOTIONS OF AN OSCILLATOR OF ONE DEGREE
OF FREEDOM; CONTROL WITH REGULATOR-POSITION
CONTROL WITHOUT SHIFTINGS OF REVERSALS.

Abstract: So-called flip-flop controls (also called "on-off-course controls") are frequently preferred to continuous controls because of their simple construction. Thus they are used also for the steering control of airplanes. Such a body possesses - even if one thinks, for instance, only of the symmetric longitudinal motion - three degrees of freedom so that a study of its motions under the influence of an intermittent control is at least lengthy. Thus, it is suggested that an investigation of the basic effect of such a control first be made on a system with one degree of freedom¹. Furthermore, we limit ourselves in the present report to the investigation of an "ideal" control where the control surface immediately obeys the command given by the "steering control function". Thus the oscillation properties of the control surface and the defects in linkage, sensing element, and mixing device are, at first, neglected. As long as the deviations from the "ideal" control may be neglected in practice, also the motion of the control

*"Über Bewegungen eines Schwingers unter dem Einfluss von Schwarz-Weiss-Steuerungen." Zentrale für wissenschaftliches Berichtswesen der Luftfahrtforschung des Generalluftzeugmeisters (ZWB), Untersuchungen und Mitteilungen Nr. 1326, Berlin, August 1, 1943.

¹In carrying out these investigations we were effectively supported by study councillor K. Scholz (DVL) and H. F. Hodapp (Vifs). In the beginning Dr. Bader (DVL) also participated in the investigations.

surface takes place at the beat of the motion of the principal system. The aim of our investigation is to obtain a survey of the influence of the system and control coefficients on the damping behavior which is to be attained².

- Outline:
1. ~~The Flip-Flop Control~~ (System Equation, ~~Steering-Control~~ Equation, Equation of motion, ~~Steering-Control~~ Function, Systems of Type A and B, Reversal Points of Type a and b)
 2. The Finite Equation of Motion, the Equation for the Control Function and Its Derivative
 3. Representation of the Course of Motion. The Generating Motion. The Phase Plane. The Plane of the Reversal Values
 4. Division of the Plane of the Reversal Points into "Intercepts" and "Limit Points" ("Starting Points" and "End Points", Respectively) and into Reversal Points a and Reversal Points b, Respectively
 5. Curves of Constant Interval Length νt_1 (Isochrones) in the Plane of the Reversal Values and their Particular Cases
 6. Energy Consideration and Periodic Solutions
 7. Comprehensive Discussion of the Possible Courses of Motion (Discussion Figures 27 and 28)

²The same problem, although within a somewhat narrower scope (only system A), has already been treated by H. Bilharz in a report entitled "Über eine gesteuerte eindimensionale Bewegung" (On a controlled one-dimensional motion), Zeitschrift für angewandte Mathematik und Mechanik, 22, pp. 206-215). The methods of investigation differ. H. Bilharz stressed particularly the mathematical representation, whereas we strove to emphasize the mechanical side of the problem. After the reports had been written independently, discussions held in common showed complete agreement of the results.

1. The Flip-Flop Control³ (System Equation, Steering-Control Equation, Equation of Motion, Steering-Control Function, Systems of Type A and B, Reversal Points of Type a and b.)

The controlled motions of a vehicle (which itself is regarded as a system of one degree of freedom) are determined by two equations: a) the equation of motion of the system ("system equation"), b) the steering-control equation. The system equation may be written, as a rule, with greater or smaller approximations, in the form

$$a'\ddot{\varphi} + b'\dot{\varphi} + c'\varphi = N\beta \quad (1.1)$$

where φ signifies the quantity characteristic for the motion (control or regulated quantity) - thus for the course control of a vehicle, for instance, the deviation from the course, β , the so-called adjustment quantity - for the course control the control-surface deflection; the factor N is occasionally denoted as "control-surface effectiveness".

³Recently a distinction in regulation technique has been made between the concepts "regulation" and "control" (Regulation Technique, Concepts and Symbols, edited by the VDI - Special Committee for Regulation Technique, outline, p. 3). A control is spoken of only when the variation in the position of the regulated quantity does not occur on the basis of a measurement of this quantity which is to be influenced; every variation of the regulated quantity based on a measurement of this quantity which is to be influenced is called regulation. In this sense every so-called "automatic control of a vehicle" also is a regulation. In this case, however, the designation control is retained, in agreement with general colloquial usage in the technique of automatic vehicle control and particularly in aviation technique. For flip-flop control the outline mentioned above suggests the designation "open-shut regulator" or "in-out regulator" (elsewhere p. 83). The "system equation" (1.1) would be called, according to these suggestions, "equation of the regulation distance", the control equation "equation of the regulator", etc. We limit ourselves here to these directives.

If the regulation of the motion is performed by a control with continuous variation of the regulated quantity, the control equation reads in general

$$\sum_{\kappa=0}^k m_{\kappa} \beta^{(\kappa)} = \sum_{\nu=0}^n a_{\nu} \varphi^{(\nu)} \quad (1.2)$$

The number of the terms in the sums depends on the number of the sensed derivatives $\varphi, \dot{\varphi}, \ddot{\varphi}, \dots$ and on the structure of the regulator (control machine) by which the number of the derivatives of β is determined.

In (1.2) for instance the equation of the simple regulator position control

$$\beta = a_0 \varphi + a_1 \dot{\varphi} \quad (1.2a)$$

or of the simple flight-velocity control

$$\dot{\beta} = a_0 \dot{\varphi} + a_1 \ddot{\varphi} \quad (1.2b)$$

is contained (with use of two sensed quantities).

By substitution of the equation (1.1) -- solved with respect to β and differentiated as often as necessary -- in the left side of (1.2), there results for the coordinate φ a homogeneous linear differential equation of higher order with constant coefficients. It describes the controlled motion under investigation. The roots of its secular equation for instance give information on the stability behavior.

In this report a so-called flip-flop control (also called on-off-course control), in contrast to a control working continuously, will be investigated. The characteristic of such a control consists in the fact that the adjustment quantity β does not vary continuously and proportionally to a linear combination of the sensed quantities $\varphi, \dot{\varphi}, \ddot{\varphi}, \dots$, but can assume only two constant values $+\beta_0$ or $-\beta_0$. The control equation reads in this case, if a

linear combination of the two quantities φ and $\dot{\varphi}$, namely $F = \rho_1\varphi + \rho_2\dot{\varphi}$, is sensed:

$$\beta = -\beta_0 \operatorname{sgn}(\rho_1\varphi + \rho_2\dot{\varphi}) \quad (1.3a)$$

or

$$\beta = +\beta_0 \operatorname{sgn}(\rho_1\varphi + \rho_2\dot{\varphi}) \quad (1.3b)$$

The function $F = \rho_1\varphi + \rho_2\dot{\varphi}$, the sign of which is in (1.3a) as well as in (1.3b) deciding the sign of the adjustment quantity β , will be called the (steering) control function below.

If the control function F passes through zero, the adjustment quantity β is, in a discontinuous manner, "reversed" from its positive value β_0 to its negative value $-\beta_0$ (or inversely). Thus the points where the function F passes through zero are the "reversal points" of the control. As long as (like in this consideration) the zero passage of the control function is decisive for the reversal, one may consider instead of

$$F = \rho_1\varphi + \rho_2\dot{\varphi}$$

the function

$$F = \varphi + \rho\dot{\varphi} \quad (1.4)$$

(where $\rho = \rho_2/\rho_1$). In this report the control function will always be used in the form (1.4).

If $\rho > 0$, the control function F shows, with respect to the deflection, a lead by the time t_v ; if $\rho < 0$, it shows a lag by t_n (cf. fig. 1).

Besides, the time is counted so that from each reversal point a count starts with $t = 0$. The values $\varphi(0) = \varphi_0$ and $\dot{\varphi}(0) = \dot{\varphi}_0$

are also called the reversal values. Because of $F(0) = 0$ the relation

$$\rho = -\dot{\varphi}_0 / \ddot{\varphi}_0 \quad (1.4a)$$

exists between them. Let the length of the interval before the next reversal point be t_1 .

A system, the control equation of which reads like (1.3a), is always denoted below as system A; a system, the control equation of which reads like (1.3b), is denoted as system B. Thus one should note that the sign of the adjustment quantity β varies with the variation of the sign of the control function F , but that two possibilities of coordination exist; in system A one has

$$\text{sgn } \beta = -\text{sgn } F$$

in system B, in contrast,

$$\text{sgn } \beta = +\text{sgn } F$$

Regardless of how the control-surface deflection may vary in the course of a motion, the correlation between the sign of β and that of F is a priori determined by the apparatus.

If one substitutes (1.3a) or (1.3b) in (1.1), the equation

$$a\ddot{\varphi} + b\dot{\varphi} + c\varphi = \mp(\text{sgn } F)N\beta_0$$

or

$$a\ddot{\varphi} + b\dot{\varphi} + c\varphi = \mp(\text{sgn } F)M \quad (1.5a)$$

results as equation of the controlled motion of the system. In it the upper sign is valid for a system of the type A, the lower for one

of the type B. $M = N\beta_0$ is the value of the control moment,

$$m = \mp(\text{sgn } F)M \quad (1.6)$$

which stands on the right side of the equation (1.5a); thus M is an essentially positive quantity.

The variation of the control function F and of the control moment with time is shown in figure 2a for a system A, in figure 2b for a system B. Furthermore, a reversal point where the control moment m jumps from its negative to the positive value is called a reversal point a, one where the control moment jumps from the positive to the negative value, a reversal point b. According to the aforesaid it is clear that in the system A at a reversal point a the control function F passes through zero from positive to negative values ($\dot{F} < 0$), at reversal point b, in contrast, from negative to positive values ($\dot{F} > 0$), whereas this correlation is reversed in the system B.

In the form (1.5a) of the equation of motion four parameters, a' , b' , c' and M , appear; furthermore all terms have the dimension of a moment (a generalized force). After division by the inertia factor (the moment of inertia) a' the equation (1.5a) assumes the form

$$\ddot{\varphi} + 2D\dot{\varphi} + \omega^2\varphi = \mp(\text{sgn } F)b \quad (1.5b)$$

the number of parameters being reduced to three, D , ω , b ; now all terms of the equation have the dimension of an angular acceleration. The significance is (as customary in oscillation theory)

$$\omega^2 = c'/a'; \quad D = b'/2\sqrt{a'c'} \quad \text{and } b = M/a'$$

The upper sign on the right side again applies to systems of the type A, the lower one to systems of the type B.

Occasionally instead of the form (1.5b) a further form of the equation of motion is of advantage; it results from (1.5b) after

division by ω^2 and reads

$$\frac{\ddot{\phi}}{\omega^2} + 2D \frac{\dot{\phi}}{\omega} + \phi = \mp(\text{sgn } F) \frac{b}{\omega^2} \quad (1.5c)$$

Here all terms have the dimension of the coordinate ϕ itself. Finally, with a new independent variable $\tau = \omega t$ due to which the derivative becomes

$$\frac{1}{\omega} \frac{d}{dt} = \frac{d}{d\tau}$$

the differential equation (1.5c) could be changed to the form

$$\phi'' + 2D\phi' + \phi = \mp(\text{sgn } F) \frac{b}{\omega^2} \quad (1.5d)$$

when the derivatives with respect to τ are denoted by dashes.

In all numerical calculations we shall further on put the parameter $b/\omega^2 = M/e'$ equal to unity. This means that the scales used for plotting the results are at our disposal. The parameter b/ω^2 signifies the deflection (angle) $\phi = \phi_d$ that the control moment M produces on the restoring spring c' . This "static" deflection ϕ_d then is the unit of the variable ϕ . Or, in other words, instead of the quantity ϕ , the new quantity $\bar{\phi} = \phi/\phi_d$ is used as the dependent variable. Thus the differential equation ultimately obtains the final form:

$$\bar{\phi}'' + 2D\bar{\phi}' + \bar{\phi} = \mp(\text{sgn } F) \quad (1.5e)$$

Of the four parameters a' , b' , c' , and M appearing in (1.5a) or the three essential parameters D , ω , and $b/\omega^2 = M/e'$ appearing in (1.5c), two parameters, ω and b/ω^2 , have been eliminated by the measures mentioned, namely, first, different count of time (or,

respectively, use of derivatives $\dot{\phi}/\omega$, $\ddot{\phi}/\omega^2$, etc.), second, having the scales of the variable ϕ and $\dot{\phi}/\omega$ at disposal (or, respectively, different count of the deflection) so that as a sole parameter the damping coefficient D remains. If nothing else is noted, the latter is put down for all numerical calculations as $D = 0.1$.

For better maintenance of the illustrative quality, however, the equation of motion is not used in the form (1.5e); one stops at the form (1.5b) or (1.5c).

Thus one calculates with the time t , using again as derivatives only quotients $\dot{\phi}/\omega$, $\ddot{\phi}/\omega^2$, etc.; one may also calculate in the coordinate ϕ itself, bearing in mind, however, that the latter is measured on a special scale, so that the unit denotes the static deflection ϕ_d .

2. The Finite Equation of Motion; the Equation for the Control Function and Its Derivative.

The differential equation (1.5b) describes the motion in sections only. As soon as the control function F changes its sign, another differential equation appears, since the disturbance term in (1.5b) changes its sign. Likewise, the integral of the equation of motion, the time equation of the motion, applies for that reason only to one section between two zero-passages of the control function F ; it reads

$$\phi = Ae^{\lambda_1 t} + Be^{\lambda_2 t} \mp (\text{sgn } F) \frac{b}{\omega^2}, \quad (2.1)$$

wherein

$$\begin{aligned} \lambda_1 &= -\delta + i\nu = \omega \left[-D + i \sqrt{1 - D^2} \right] \\ \lambda_2 &= -\delta - i\nu = \omega \left[-D - i \sqrt{1 - D^2} \right] \end{aligned} \quad (2.2)$$

with $\delta = D\omega$ and $\nu = \omega \sqrt{1 - D^2}$. It has already been mentioned that

the count of the time in each interval is to start from zero. The time $t = 0$ denotes the beginning of an interval; a reversal point is located there. The derivative of (2.1) reads:

$$\dot{\varphi} = \lambda_1 A e^{\lambda_1 t} + \lambda_2 B e^{\lambda_2 t} \quad (2.3)$$

Since for attainment of a real solution the integration constants A and B must be conjugate-complex, they may be written

$$A = C e^{i\epsilon}, \quad B = C e^{-i\epsilon}$$

therewith one has obtained for (2.1) and (2.3)

$$\varphi = C \left[e^{\lambda_1 t + i\epsilon} + e^{\lambda_2 t - i\epsilon} \right] + (\text{sgn } F) \frac{b}{\omega^2} \quad (2.1a)$$

and

$$\dot{\varphi} = C \left[\lambda_1 e^{\lambda_1 t + i\epsilon} + \lambda_2 e^{\lambda_2 t - i\epsilon} \right] \quad (2.3a)$$

A further useful form of the equation (2.3a) is obtained by expressing the conjugate-complex factors λ_1 and λ_2 which, because of $\sqrt{\delta^2 + \nu^2} = \omega$ have the value ω , by this magnitude and the argument $\mp\sigma$:

$$\lambda_1 = \omega e^{i\sigma} \quad \lambda_2 = \omega e^{-i\sigma} \quad (2.4)$$

inversely, one then has

$$\cos \sigma = -\frac{\delta}{\omega} = -D; \quad \sin \sigma = \frac{\nu}{\omega} = \sqrt{1 - D^2} \quad (2.5)$$

Thus one obtains instead of (2.3a)

$$\frac{\dot{\phi}}{\omega} = C \left[e^{\lambda_1 t + i(\epsilon + \sigma)} + e^{\lambda_2 t - i(\epsilon + \sigma)} \right] \quad (2.3b)$$

Analogously, the expression for the control function $F = \phi + \rho \dot{\phi}$ may be rewritten. Using (2.1a) and (2.3b) one achieves, first

$$F = C \left[e^{\lambda_1 t + i\epsilon} (1 + \omega \rho e^{i\sigma}) + e^{\lambda_2 t - i\epsilon} (1 + \omega \rho e^{-i\sigma}) \right] + (\operatorname{sgn} F) \frac{b}{\omega^2} \quad (2.6)$$

If one expresses the conjugate-complex factors (in parentheses) by their magnitude C' and their argument $\pm\tau$,

$$1 + \omega \rho e^{i\sigma} = C' e^{i\tau}; \quad 1 + \omega \rho e^{-i\sigma} = C' e^{-i\tau} \quad (2.7)$$

one obtains

$$F = CC' \left[e^{\lambda_1 t + i(\epsilon + \tau)} + e^{\lambda_2 t - i(\epsilon + \tau)} \right] + (\operatorname{sgn} F) \frac{b}{\omega^2} \quad (2.8a)$$

therein the magnitude is

$$C' = \sqrt{1 - 2D\omega\rho + \omega^2\rho^2} \quad (2.8a)$$

and the argument

$$\tau = \arctan \frac{\omega\rho \sqrt{1 - D^2}}{1 - D\omega\rho} \quad (2.8b)$$

The derivative \dot{F} of the control function shows the factors λ_1 and λ_2 before the e -functions. They may again be expressed according to (2.4) by magnitude and argument. Thus one obtains

$$\frac{\dot{F}}{\omega} = CC' \left[e^{\lambda_1 t + i(\epsilon + \sigma + \tau)} + e^{\lambda_2 t - i(\epsilon + \sigma + \tau)} \right] \quad (2.9)$$

Finally we indicate how the integration constants C and ϵ are related to the initial values φ_0 and $\dot{\varphi}_0/\omega$:

$$C = \frac{1}{2} \frac{1}{\sqrt{1 - D^2}} \sqrt{\left[\varphi_0 \pm (\operatorname{sgn} F) \frac{b}{\omega^2} \right]^2 + 2D \frac{\dot{\varphi}_0}{\omega} \left[\varphi_0 \pm (\operatorname{sgn} F) \frac{b}{\omega^2} \right] + \left(\frac{\dot{\varphi}_0}{\omega} \right)^2} \quad (2.10)$$

$$\tan \epsilon = - \frac{1}{\sqrt{1 - D^2}} \left[\frac{\dot{\varphi}_0/\omega}{\varphi_0 \pm (\operatorname{sgn} F) \frac{b}{\omega^2}} + D \right] \quad a)$$

$$\sin \epsilon = - \frac{1}{2C \sqrt{1 - D^2}} \left[\frac{\dot{\varphi}_0}{\omega} + D \left(\varphi_0 \pm (\operatorname{sgn} F) \frac{b}{\omega^2} \right) \right] \quad b) \quad (2.11)$$

$$\cos \epsilon = \frac{1}{2C} \left[\varphi_0 \pm (\operatorname{sgn} F) \frac{b}{\omega^2} \right] \quad c)$$

The equations (2.1a), (2.3b), (2.6a) and (2.9) show the functions $\varphi(t)$, $\frac{\dot{\varphi}(t)}{\omega}$, $F(t)$, and $\frac{\dot{F}(t)}{\omega}$. We state them once more, splitting the coefficients λ_1 and λ_2 into their real and imaginary parts:

$$\begin{aligned}
\varphi(t) &= C e^{-\delta t} \left[e^{i(\nu t + \epsilon)} + e^{-i(\nu t + \epsilon)} \right] \mp (\operatorname{sgn} F) \frac{b}{\omega^2} \\
\frac{\dot{\varphi}(t)}{\omega} &= C e^{-\delta t} \left[e^{i(\nu t + \epsilon + \sigma)} + e^{-i(\nu t + \epsilon + \sigma)} \right] \\
F(t) &= C C' e^{-\delta t} \left[e^{i(\nu t + \epsilon + \tau)} + e^{-i(\nu t + \epsilon + \tau)} \right] \mp (\operatorname{sgn} F) \frac{b}{\omega^2} \\
\frac{\dot{F}(t)}{\omega} &= C C' e^{-\delta t} \left[e^{i(\nu t + \epsilon + \sigma + \tau)} + e^{-i(\nu t + \epsilon + \sigma + \tau)} \right]
\end{aligned} \tag{2.12}$$

The brackets may also be written in real form according to the de Moivre theorem:

$$\begin{aligned}
\varphi(t) &= 2C e^{-\delta t} \cos(\nu t + \epsilon) \mp (\operatorname{sgn} F) \frac{b}{\omega^2} \\
\frac{\dot{\varphi}(t)}{\omega} &= 2C e^{-\delta t} \cos(\nu t + \epsilon + \sigma) \\
F(t) &= 2C C' e^{-\delta t} \cos(\nu t + \epsilon + \tau) \mp (\operatorname{sgn} F) \frac{b}{\omega^2} \\
\frac{\dot{F}(t)}{\omega} &= 2C C' e^{-\delta t} \cos(\nu t + \epsilon + \sigma + \tau)
\end{aligned} \tag{2.13}$$

In order to prepare for later applications, we indicate explicitly what form the third equation of (2.13) assumes if the integration constants are expressed by the initial values:

$$F(t) = \pm (\text{sgn } F) \frac{b}{\omega^2} e^{-\delta t} \cos \nu t$$

$$+ \frac{1}{\sqrt{1-D^2}} \left[\frac{\dot{\phi}_0}{\omega} (1 - D\omega\phi) + \left(\phi_0 \pm (\text{sgn } F) \frac{b}{\omega^2} \right) (D-\omega\phi) \right] e^{-\delta t} \sin \nu t \quad (2.14)$$

$$\mp (\text{sgn } F) \frac{b}{\omega^2}$$

At the reversal points $t = 0$ one obtains

$$F(0) = 0$$

and

$$\frac{\dot{F}(0)}{\omega} = \frac{1}{\dot{\phi}_0/\omega} \left[\left(\frac{\dot{\phi}_0}{\omega} \right)^2 + 2D\phi_0 \frac{\dot{\phi}_0}{\omega} \pm (\text{sgn } F) \frac{b}{\omega^2} \phi_0 + \phi_0^2 \right]$$

$$= \frac{1}{\dot{\phi}_0/\omega} K \quad (2.15)$$

as one can recognize from $\ddot{F} = \ddot{\phi} + \rho\ddot{\phi}$, using the differential equation (1.5b).

3. Representation of the Course of the Motion. The Generating Motion. The Phase Plane. The Plane of the Reversal Values.

In order to represent the course of a motion one may use various expedients. The first, conventional one, consists of plotting the variable ϕ as a function of the time; thus one obtains the deflection-time-diagram (ϕ - t -diagram). The ordinate of this diagram, the deflection itself, may be obtained for special classes of motions (to which also belong those of interest here) as the projection of a "generating" motion. The generating motion of an harmonic oscillation is, for instance, a circular motion with constant angular velocity ω , a fact which is well known and often put to use.

The functions $\varphi(t)$, $\frac{\dot{\varphi}(t)}{\omega}$ and $F(t)$ (cf. the equations (2.13)) all show the same damping factor; thus they may all be written in the form

$$y = Ce^{-\delta t} \cos(vt + a) + y_0 \quad (3.1)$$

Such functions may, however, be represented as the projection of the motion of a point which moves with constant angular velocity v on a spiral (displaced from the zero point by the distance y_0). Here the motion along the spiral is the "generating" motion. The spirals which lead to the functions φ , $\dot{\varphi}/\omega$, F , and \dot{F}/ω are congruent. They differ only by the angle at which the starting point of the spiral lies (or, respectively, by which the spiral is rotated compared to an initial position).

The facts mentioned here are readily understandable. If one takes in addition to

$$y = Ce^{-\delta t} \cos(vt + a) \quad (3.2a)$$

the function

$$x = Ce^{-\delta t} \sin(vt + a) \quad (3.2b)$$

and forms either

$$z = y + ix \quad \text{or} \quad r = \sqrt{x^2 + y^2}$$

one arrives at the complex or real form of the equation of a logarithmic spiral in the form

$$z = Ce^{(-\delta + iv)t + ia} \quad (3.3a)$$

or

$$r = Ce^{-\frac{\delta}{v}(vt)} \quad (3.3b)$$

Thus $y(t)$ is one, $x(t)$ the other rectangular projection of the motion of a point on the spiral.

In the course of our calculations we have made wide use of this expedient of the generating motion on the spiral. The solution of transcendental equations which contain expressions of the form (3.1), for instance $F(t) = 0$, is thus possible in a simple manner.

The phase curve (curve in the phase plane) forms another means for following the course of a motion. Coordinates of the phase plane are the variables φ and $\dot{\varphi}$ or multiples of them; for using variables of equal dimension we shall always select φ and $\dot{\varphi}/\omega$. The time t plays for such "phase curves" the role of a parameter; to each point of a phase curve there corresponds a constant value t .⁴ By differentiating the equation $\varphi = \varphi(t)$ one obtains the second equation, $\dot{\varphi}/\omega = \dot{\varphi}(t)/\omega$; both equations together form the representation by parameters of the phase curve.

For a harmonic oscillation the equation of the deflection-time-diagram reads

$$\varphi = A \cos(\omega t + \alpha) \quad (3.4)$$

the diagram represents a cosine-line. For the curve in the phase plane the parametric representation is

$$\varphi = A \cos(\omega t + \alpha) \quad (3.4a)$$

$$\frac{\dot{\varphi}}{\omega} = -A \sin(\omega t + \alpha) \quad (3.4b)$$

⁴Phase denotes in general physics any complex of the quantities determinant for a state (pressure, temperature...). A state of motion is completely determined by the coordinate and all its derivatives with respect to time; φ , $\dot{\varphi}$, $\ddot{\varphi}$, determine the phase of a state of motion (they determine a point in the phase space), φ and $\dot{\varphi}$ are two of the quantities determinant for the phase of a motion. They are the coordinates of the phase plane. In a problem "of the second order" the phase is already completely determined by two quantities φ and $\dot{\varphi}$.

The equation of the curve, resulting by elimination of t , reads

$$\varphi^2 + \left(\frac{\dot{\varphi}}{\omega}\right)^2 = A^2 \quad (3.4c)$$

it is a circle of the radius A . Thus to a harmonic oscillation a circle corresponds as generating motion as well as phase curve (fig. 3).

If a controlled motion of the kind named in sections 1 and 2 is described in the phase plane, all the reversal points for a certain system lie on a straight line, the "line of reversals". Reversals are made when $F = \varphi + \rho\dot{\varphi} = 0$; however, since in that case the time count also starts anew, $\varphi = \varphi_0$ and $\dot{\varphi} = \dot{\varphi}_0$. Thus one obtains

$$\rho = -\frac{\varphi_0}{\dot{\varphi}_0}$$

or

$$\tan \alpha = +\frac{\varphi_0}{\dot{\varphi}_0/\omega} = -\omega\rho \quad (3.5)$$

Thus the reversal points lie on the straight line, the slope $\tan \alpha$ of which has the value $-\omega\rho$. (If $\omega\rho > 0$, the straight lines go through the second and fourth quadrant, if $\omega\rho < 0$, through the first and third quadrant).

The phase curves are similar to logarithmic spirals, but do not coincide exactly with them. Real logarithmic spirals as phase curves are obtained, however, if one places into the phase plane instead of a rectangular (rectilinear) coordinate system an oblique-angled one.

This can easily be proved with the aid of figure 4. Let the angle between the negative abscissa-axis ($-\dot{\varphi}/\omega$ -axis) and the positive ordinate axis (φ -axis) be the obtuse angle σ as it is determined as function of D by the equations (2.5). If one applies the cosine theorem to the shaded triangle, one obtains

$$r^2 = \left[\varphi \pm (\operatorname{sgn} F) \frac{b}{\omega^2} \right]^2 + \left(\frac{\dot{\varphi}}{\omega} \right)^2 - 2 \left[\varphi \pm (\operatorname{sgn} F) \frac{b}{\omega^2} \right] \frac{\dot{\varphi}}{\omega} \cos \sigma \quad (3.6a)$$

If φ and $\dot{\varphi}/\omega$ therein are expressed as functions of the time according to the equations (2.13), there results

$$r^2 = 4C^2 e^{-2\delta t} \left[\cos^2(vt + \epsilon) + \cos^2(vt + \epsilon + \sigma) - 2 \cos(vt + \epsilon) \cos(vt + \epsilon + \sigma) \cos \sigma \right] \quad (3.6b)$$

Splitting of $\cos[(vt + \epsilon) + \sigma]$ according to the addition theorem then yields

$$r = (2C \sin \sigma) e^{-\frac{\delta}{v} (vt)} \quad (3.7)$$

thus the equation of a logarithmic spiral.

The lines of reversals, which have, as shown above, in a rectangular coordinate system the slope $\tan \alpha = -\omega p$, have in the oblique-angled coordinate system mentioned the slope $\tan \alpha = -\tan \tau$. This may be recognized with the aid of figure 4. For the angle α plotted there one finds

$$\tan \alpha = \frac{\varphi_0 \sin \sigma}{\dot{\varphi}_0/\omega - \varphi_0 \cos \sigma} = -\frac{\omega p \sin \sigma}{1 + \omega p \cos \sigma}$$

Hence, then results, with use of the equations (2.5) and (2.8)

$$\tan \alpha = -\frac{\omega p \sqrt{1 - D^2}}{1 - D \omega p} = -\tan \tau \quad (3.8)$$

thus

$$\alpha = \pi - \tau$$

From equation (3.7) also follows that the angle subtended at the point of convergence of a spiral between the rays toward any two phase points has the value vt ; thus it is a measure for the time required by the motion. Hence it follows directly that the angle between the rays toward the points of reversal has the value vt_1 at the beginning and at the end of the interval.

Since for system A the point of convergence of the spiral arc always lies on the side of the line of reversals turned away from the arc, for system B, however, on the side turned toward the arc, it follows immediately (cf. fig. 5) that for system A the angle is $\nu t_1 \leq \pi$, for system B $\nu t_1 \geq \pi$ (cf. also 5).

The fact that the phase curve in the oblique-angled system is a logarithmic spiral greatly facilitates following the course of the motion even through many intervals. One draws several windings of a

spiral $r = e^{-\frac{\delta}{\nu}(\nu t)}$. Moreover one draws on a transparent paper provided with oblique-angled coordinate lines φ and $\dot{\varphi}/\omega$ (cf. fig. 6) the points $\pm b/\omega^2$ and the line of reversals. The line of reversals - for the points of which the control function is $F(t) = 0$ - then divides the phase plane into values pertaining to positive $F(t)$ and into values pertaining to negative $F(t)$. Let the starting point of the motion be P_a (cf. fig. 6). The control is to be handled

according to system A. The control function $F = \varphi + \rho \dot{\varphi}$ has at the point P_a for the positive ρ a positive sign. Thus the

point $-b/\omega^2$ must be fixed to the point of convergence of the spiral, and the spiral must be rotated so that it passes through P_a . The

first point of intersection of the spiral with the line of reversals then is the initial reversal value S_0 . At the reversal point S_0 ,

F changes its sign. One must, therefore, in continuing the

construction of the phase curve, fix the point b/ω^2 to the point of convergence of the spiral; one may now draw the phase curve from S_0 to the next reversal point S_1 . In this manner the entire course of the motion may be quickly constructed.

In constructing such phase curves one observes easily two peculiarities which may occur in the course of the motion. First, in the proximity of the zero point of the phase plane there may exist reversal points starting from which the phase curve does not again meet the line of reversals (fig. 7). Such a "missing" of the line of reversals may occur only in the case B, not in the case A, however strong the "damping" may be, for in the case A the point of convergence of the spiral lies "behind" the line of reversals. If the phase curve does not again meet the line of reversals, no further reversal takes place. Starting from the last reversal point the same functions $\varphi(t)$ and $(\dot{\varphi}/\omega)(t)$ continually describe the motion. The motion is a damped oscillation convergent toward the

point $\varphi_d = \pm \frac{b}{\omega^2}$. This case will be treated more thoroughly; one speaks there of a "rest" of the control, the last reversal point being called "rest point".

The second peculiarity of the course of motion can also be easily understood from the phase pattern (cf. fig. 8). We have for instance a system A; we come from a reversal point S_1 , the distance of which from the zero point has the order of magnitude b/ω^2 , to the farther S_2 without having intersected the $\dot{\varphi}/\omega$ -axis, and remain therefore at the left of the zero point on the line of reversals. If we now continue with the construction in the customary manner, we find that the phase curve attempts to run further on the same side of the line of reversals, that is, the control function attained in S_2 the value 0, but did not, however, change its sign. Since, however, the "zero passage" is the condition for a reversal, the behavior of the control surface is not determined here. Only if for instance the reversal should occur somewhat after the zero passage of F (lagging of the control surface), we could construct further (cf. fig. 9). One can see that the motion then assumes very high frequency and that its deflections decrease. The cases leading to such an undefined behavior of the "ideal" control are treated in detail in sections 4 and 7.

In the further course of the investigation in addition to the phase plane another plane, showing a close relationship to it, will be used frequently. In the phase plane with the coordinates φ and $\dot{\varphi}/\omega$ the motion may still be described continuously; a phase curve forms the sequence of all points of state. However, in our further investigations we shall often be content to consider instead of the succession of all points of state the sequence of the reversal points, that is, of those values $\varphi = \varphi_0$ and $\dot{\varphi}/\omega = \dot{\varphi}_0/\omega$ which exist at the moment when the zero passage of the function F takes place. Thus from the points of the phase plane the reversal points $\varphi_0, \dot{\varphi}_0/\omega$ (which lie on a straight line, the line of reversals) are singled out and used for characterization of the motion. If one again forms a plane from the lines of reversals pertaining to all kinds of systems, one obtains the $\varphi_0 - \dot{\varphi}_0/\omega$ -plane or plane of the reversal values. In this plane of reversal values the motion of a system is indicated by a sequence of points in isolated location. No further information is obtained concerning the course of the motion between the reversal points. The essential characteristics of the motion, however, - above all, the damping, the increase of the

amplitude, etc. - may be recognized even in such a representation, for the phase curves are logarithmic spirals with a point of convergence lying either at the point b/ω^2 or $-b/\omega^2$. Thus one is certain that the motion between two reversal points does not show any peculiarities. The damping of the motion is assured when the sequence of the reversal points converges toward zero.

However, whereas we placed an oblique-angled system of coordinates into the phase plane, we shall always place a rectangular one into the plane of reversals. The lines of reversals then again obtain the slopes $\tan \alpha = -\omega\rho$. This plane of the reversal points, the $\phi_0 - \dot{\phi}_0/\omega$ - plane, will be discussed in great detail later on.

An example is shown for illustration: Of the motion of a system represented by the phase curves of figure 10 (here drawn in Cartesian coordinates) only the reversal points appear in the $\phi_0 - \dot{\phi}_0/\omega$ - plane (fig. 11). For the prescribed system with constant values ρ and ω the reversal points lie on the straight line of the slope $\tan \alpha = -\omega\rho$. The points on straight lines of different slope pertain to other systems. However, it must be noted that the difference in the distances of two reversal points from the zero point is not yet a measure for the rate of damping. This difference measures the damping per interval. In order to arrive at a damping velocity, the length of the intervals also must be taken into consideration (cf. section 6).

A periodic motion is, for instance, represented by two reversal points repeated successively again and again (fig. 12). However, these points need not by any means denote the maximum deflection which occurs during the motion; on the contrary, maximum deflections are reversal points only when $\dot{\phi}_0/\omega = 0$; such points lie on the ordinate axis.

4. Division of the Plane of the Reversal Points into "Intercepts"

and "Limit Points" ("Starting Points" and "End Points",

Respectively), and into Reversal Points a and

Reversal Points b, Respectively

We start out considering the variation with time of the functions $F(t)$ and $m(t)$, as it is represented in figure 2. The

zeros of the control function $F(t)$ denote the reversal points. At these zero points the function $F = \phi + \omega\phi' (\phi'/\omega)$ always shows a break, because ϕ''/ω^2 jumps (as follows from the differential equation, for instance (1.5b)). The jump of ϕ''/ω^2 at a reversal point is

$$\frac{\phi''}{\omega^2} (+0) - \frac{\phi''}{\omega^2} (-0) = + \frac{2b}{\omega^2} \quad \text{for reversal points } a \quad (4.1a)$$

$$\frac{\phi''}{\omega^2} (+0) - \frac{\phi''}{\omega^2} (-0) = - \frac{2b}{\omega^2} \quad \text{for reversal points } b$$

Because of the relation following from the definition of F

$$\frac{\dot{F}}{\omega} = \frac{\dot{\phi}}{\omega} + \omega\phi''$$

there results, therefore, for the jump of \dot{F}/ω , which characterizes the break of the function F , at the reversal points a the value

$$\frac{\dot{F}}{\omega} (+0) - \frac{\dot{F}}{\omega} (-0) = 2\omega\phi'' \frac{b}{\omega^2}$$

at the reversal points b

(4.1b)

$$\frac{\dot{F}}{\omega} (+0) - \frac{\dot{F}}{\omega} (-0) = - 2\omega\phi'' \frac{b}{\omega^2}$$

A positive value of the jump (right side of (4.1b)) for decreasing function F signifies a "breaking away from the perpendicular" (fig. 13a), for increasing function F a "breaking toward the perpendicular" if designations are used as they are customary in optics for the consideration of the ray path at the boundary of two media of different density. In contrast, a negative value of the jump signifies for decreasing function F a "breaking toward the perpendicular" (fig. 13b), for increasing function F a "breaking away from the perpendicular". "Breaking toward the perpendicular" increases the amount of the slope, "breaking away from the perpendicular" reduces it.

If one now considers what was said about the sign of the jump in the equations (4.1b), together with what was said in section 1 about the occurrence of reversal points a and b in the systems A and B, the following possibilities result:

Table 4/1

	System A	System B
$\omega p > 0$	breaking away from the perpendicular	breaking toward the perpendicular
$\omega p < 0$	breaking toward the perpendicular	breaking away from the perpendicular

The correlations are independent of the "character" of the reversal points, whether a or b .

As an example, we comment on the first case: If $\omega p > 0$, the jump values are, according to (4.1b), positive at reversal points a , negative at reversal points b . However, in the system A the function F decreases at reversal points a (where the jump value is positive) so that there a breaking away from the perpendicular occurs; at reversal points b (where the jump value is negative) it increases so that again a breaking away from the perpendicular occurs. The remaining three cases may be discussed correspondingly.

From the fact that the break (change in slope) which the function F shows at the reversal points has a finite value, it follows that in those cases (that is, in quadrants where a breaking "away from the perpendicular" occurs) "directions of incidence" exist to which no "directions of refraction" on the other side of the time axis correspond; thus the function F does not again pass zero and, therefore, no more reversals occur. These cases correspond to the total reflections of the optical analogy (shaded angle section in fig. 13a). Inversely, in case of breaking "toward the perpendicular" "directions of refraction" exist to which no "directions of incidence" belong (fig. 13b). Thus in both cases no more reversal occurs; the zero points (reversal points) of the function F are no longer "intercepts" (of the controlled motion), but "limit points" where the

motion either must end ("end points") or may start ("starting points"). The directions of incidence without directions of refraction as well as the directions of refraction without directions of incidence lie in the angle sections shaded in figures 13a and b. The limitation of the angle space is given by the condition that for the direction of incidence or refraction $\bar{F} = 0$. If one substitutes for \bar{F} the expression according to (2.15), one obtains

$$\left(\frac{\dot{\phi}_0}{\omega}\right)^2 + 2D \frac{\bar{\phi}_0}{\omega} \phi_0 \pm (\text{sgn } F) \frac{b}{\omega^2} \phi_0 + \phi_0^2 = 0 \quad (4.2)$$

When $D < 1$, equation (4.2) represents for system A (upper sign) as well as for system B (lower sign) two ellipses in the plane of reversals, since $\text{sgn } F$ may be either 1 or -1 (fig. 14). If the third term has the negative sign, the equation describes the upper ellipse, if it has the positive sign, it describes the lower ellipse. On the boundary of the ellipses lie the "limiting cases of total reflection"; within the ellipse lie limit points; outside of it, intercepts.

Whether the limit points lying in the interior of the ellipse are end points or starting points may be decided for instance in the following manner: For end points, breaking away from the perpendicular always occurs; for starting points, breaking toward the perpendicular. The table 4/1, which classifies the quadrants in this respect, gives, therefore, information also about the question where end points and where starting points lie in the space inside of the ellipses. The results of this consideration are plotted in figures 14a and 14b. The ellipses represented by equation (4.2) are occasionally called the "small limiting ellipses" to distinguish them from other curves to be discussed later.

As explained in section 1, the reversal points a and b differ by the sign which the function F shows at the point zero. The function $\bar{F}(0)/\omega$ is given in (2.15). Thus $\bar{F}(0)/\omega$ is positive as long as both factors, $\dot{\phi}_0/\omega$ and the brackets K , show the same sign; \bar{F}/ω is negative when the factors have different signs. The limiting case $\bar{F}/\omega = 0$ occurs when the brackets vanish; the significance of this case was discussed just now. The equation $K = 0$ represents the "small limiting ellipses" discussed above. Reversal values ϕ_0 , $\dot{\phi}_0/\omega$ from the space inside of the ellipses are

for $K < 0$; reversal values from the space outside of the ellipses are for $K > 0$.

We inquire first: What regions of the $\varphi_0 - \dot{\varphi}_0/\omega$ - plane, for the system A, contain reversal points a. To system A pertains the upper sign of the third term in K (2.15). Furthermore, one has for every reversal point a (cf. fig. 2a) in the system A (after the reversal) $\text{sgn } F = -1$. Therewith the term in question assumes the negative sign; thus of the two "small limiting ellipses" only the upper one is considered. Reversal points a in the system A require, moreover, $\dot{F} < 0$, thus different signs for the factors $\dot{\varphi}_0/\omega$ and K.

In the right semiplane where $\dot{\varphi}_0/\omega > 0$, K must be less than 0; the reversal points a lie, therefore, in the space inside of the (upper) ellipse. In the left semiplane where $\dot{\varphi}_0/\omega < 0$, K must be greater than 0; the reversal points a lie in the space outside of the same ellipse (cf. fig. 15a).

If one investigates reversal points b, one obtains $\text{sgn } F = 1$, the third term in K becomes positive, the lower one of the limiting ellipses is of importance. Since \dot{F}/ω must now be greater than 0, the reversal points b lie in the right semiplane in the space outside, in the left semiplane in the space inside of the (lower) ellipse (fig. 15b).

The investigation of system B is similarly found. Here the third term of K (2.15) has the lower sign. For reversal points a, $\text{sign } F = 1$, so that the negative sign for the term remains; by this fact (as for reversal points a of system A) the upper ellipse applies. The reversal points a require that $\dot{F}/\omega > 0$; they lie in the right semiplane in the space outside and in the left semiplane in the space inside the (upper) ellipse. (Fig. 15c.) For reversal points b, because the sign $F = -1$, the lower ellipse is of importance; since $\dot{F}/\omega < 0$, the reversal points b lie in the right semiplane in the space inside and in the left semiplane in the space outside the (lower) ellipse. (Fig. 15d.)

If one now visualizes the figures 15a and 15b of system A superimposed, one can see that within the ellipses at the upper left and lower right no reversal points, either a or b, are situated.

These points we recognized in section 3 as end points of the motion. In contrast, the starting points which lie within the ellipses at the upper right and lower left are included in the consideration, since the definition of the reversal points makes use of the sign of the function F after passing zero, and this sign actually is explained for starting points. It even becomes evident that these last-named regions are doubly covered over when figures 15a and 15b are superimposed, and thus contain points of type a as well as of type b . The motion may, therefore, begin from all starting points in two different manners. Outside of the ellipses lie intercepts which are now unequivocally divided into reversal points a and b .

5. Curves of Constant Interval-Length vt_1 in
the Plane of the Reversal Values (Isochrones)
and Their Particular Cases

The length of the intervals vt is different for the separate reversal points. We investigate here the geometrical loci of all the reversal points for which the following interval is of equal length. It can be shown that these geometrical loci are ellipses.

From the condition that

$$F(t_1) = 0 \quad (5.1)$$

follows with the aid of equation (2.14a) after a few transformations which may easily be supplemented

$$\begin{aligned} \left(\frac{\dot{\phi}_0}{\omega}\right)^2 \pm (\operatorname{sgn} F) \frac{b}{\omega^2} \left[D + \sqrt{1 - D^2} \frac{\cos vt_1 - e^{\frac{\delta}{v} vt_1}}{\sin vt_1} \right] \frac{\dot{\phi}_0}{\omega} \\ + 2D\phi_0 \frac{\dot{\phi}_0}{\omega} \pm (\operatorname{sgn} F) \frac{b}{\omega^2} \phi_0 + \phi_0^2 = 0 \end{aligned} \quad (5.2)$$

For $D < 1$ this is the equation of an ellipse in the plane of reversals with vt_1 as a parameter.

The numerical investigation shows that for the system A only ellipses may appear for which $vt_1 \leq \pi$, for the system B only ellipses for which $vt_1 \geq \pi$. This fact is immediately understandable if one considers that in the system A the control moment essentially strengthens and in the system B essentially weakens the restoring moment. A glance at the phase curves of section 3 (fig. 5) also shows that the length of the interval compared to an uncontrolled motion is diminished in the case A, increased in the case B.

In section 3 it had also been recognized that, if the length of the interval t_1 is increased, the phase curves finally no longer intersect the line of reversals so that no further reversal occurs. The last reversal point is called "rest point". In the deflection-time-diagram the limiting case of such a motion appears as indicated in figure 16. This limiting case is characterized by the additional condition (besides (5.1))

$$\dot{F}(t_1) = 0 \quad (5.3)$$

This equation (5.3) singles out solely one ellipse from the family of ellipses and determines the parameter t_1 .

Using equation (2.14) and its derivative, one obtains from the conditions (5.1) and (5.3) the transcendental relation

$$\cos vt_1 - \frac{\delta}{v} \sin vt_1 - e^{-\frac{\delta}{v}(vt_1)} = 0 \quad (5.4)$$

for the parameter vt_1 . It is satisfied

- 1) for $vt_1 = 0$
- 2) for a value different from zero $vt_1 = \bar{vt}_1$.

This last value is the one that corresponds to the "limiting rest condition". Thus an ellipse is determined in the plane of reversals by $\nu t_1 = \nu \bar{t}_1$; it is occasionally called the "large limiting ellipse".

The maximum interval length following from equation (5.4) has, for $D = 0.1$, the value

$$\nu \bar{t}_1 \approx 300^\circ$$

Such rest points can occur only in the system B. Besides, it can easily be seen that the "small limiting ellipses" mentioned in section 4 also belong to the family of ellipses of constant interval-length. Their parameter has the value $\nu t_1 = 0$; one recognizes at once that one obtains from equation (5.2), by performing there the limiting process $\nu t_1 \rightarrow 0$, the equation (4.2).

As mentioned at the beginning, the amount of damping is given the value $D = 0.1$ in all numerical calculations. All figures are drawn accordingly. The figures 17a and 17b show (for reversal points a) how the small and large limiting ellipses appear when D assumes other values. The curves are plotted for $D = 0.1$ and $D = 0.5$. In any case the curves go through the points 0 and 1. For $D = 0$ the small, and likewise the large, limiting ellipse turns into the circle through the points 0 and 1 symmetrical to the ϕ_0 -axis.

If one places — as we did — into the plane of the reversal values a Cartesian coordinate system, the curves of constant interval-length t_1 are ellipses. Without explicitly carrying out the transformation we want to note that all these ellipses become circles if one applies instead of the rectangular the oblique-angled coordinate system that was used in section 3 for the phase planes (where the negative ϕ/ω -axis forms with the positive ϕ -axis the angle σ). The equation (5.2) then is the equation of the family of these circles; the angle σ appears if one introduces it according to (2.5) instead of the amount of damping D .

6. Energy Consideration and Periodic Solutions

For the present problem the flip-flop control has the purpose of achieving a much stronger damping of the motion than attained by means of the existing damping force alone. However, in such a

controlled system not only damping, but also periodic and amplitude-increasing motions are possible since the control moment may not only remove energy from the system but may also supply energy to it.

In order to obtain a survey of where in the $\varphi_0 - \dot{\varphi}_0/\omega$ - plane lie the reversal values for damping and where for periodic and amplitude-increasing motions, we inquire first about those points in the plane of reversal values from which motions start that are deprived of energy or supplied with it by the control moment.

The energy supplied to the system by the control moment m is given by the expression

$$U_z = \int_0^{t_1} m \dot{\varphi} dt \quad (6.1)$$

With

$$m = \mp (\operatorname{sgn} F) M \quad (6.2)$$

(the upper sign again being valid for systems A, the lower for systems B) it becomes

$$U_z = \mp (\operatorname{sgn} F) M (\varphi_1 - \varphi_0) \quad (6.3a)$$

We now investigate what sign this expression has for reversal points in the individual quadrants. We consider first system A. In section 4 we determined that for system A reversal points a lie in the second and third and reversal points b in the first and fourth quadrants. With reversal points a in system A $\operatorname{sgn} F = -1$; with reversal points b $\operatorname{sgn} F = 1$. Under the further assumption that the course of the motion is always such that φ_0 and φ_1 lie on different sides of the zero point⁵, the factor $(\varphi_1 - \varphi_0)$ becomes

⁵For intervals leading to end points or starting from starting points, φ_0 and φ_1 may happen to have the same sign. In that case one must consider them separately; however, the result will again be the one shown in table 6/1.

negative for motions starting from reversal points in the first and second quadrant, positive for motions starting from reversal points in the third and fourth quadrant.

Taking these facts into consideration one finds the first column of the following table.

Table 6/1

sgn U_z

Motion starts from a reversal point in	System A	System B
1. quadrant	1	-1
2. "	-1	1
3. "	1	-1
4. "	-1	1

For the system B first the first sign in (6.3a) is reversed. Furthermore the correlations of (sgn F) to the reversal points a and b are interchanged; however, since the reversal points a and b simultaneously vary their position in the quadrants, the correlation of (sgn F) to the quadrants is maintained. Likewise the correlation of $\text{sgn } (\varphi_1 - \varphi_0)$ to the quadrants remains intact. That means the sign of U_z in system B is, for all quadrants, the opposite of that in system A (column 2 of the table 6/1).

Where $\text{sgn } U_z = 1$, energy is supplied to the system by the control moment, where $\text{sgn } U_z = -1$, energy is removed. However, where energy is removed by the control moment as well as by the damping moment, the controlled motion certainly is damped even more strongly than the uncontrolled one. Where, on the other hand, energy is added, it is damped less and the amplitude may even become constant or increase.

The motions starting from reversal points in those quadrants where the control moment supplies energy must now be considered more thoroughly in order to determine the position of the reversal points from whence start periodic, damped, or amplitude-increasing motions.

The energy supplied by the control moment is given by equation (6.3a); it may be put in the form

$$U_z = c' \left[\mp (\text{sgn } F) \frac{b}{\omega^2} (\varphi_1 - \varphi_0) \right] \quad (6.3b)$$

For the energy consumed by the damping force one has

$$U_v = \int_0^{t_1} (b' \dot{\phi}) \dot{\phi} dt = a' 2\delta \int_0^{t_1} \dot{\phi}^2 dt \quad (6.4a)$$

If one introduces $\dot{\phi}$ according to equation (2.3b) and integrates, one obtains

$$U_v = c' D \omega^2 \left[e^{2\lambda_1 t_1 + (2\epsilon + \sigma)i} + e^{2\lambda_2 t_1 - (2\epsilon + \sigma)i} - e^{(2\epsilon + \sigma)i} - e^{-(2\epsilon + \sigma)i} - \frac{2}{D} e^{-2\delta t_1} + \frac{2}{D} \right] \quad (6.4b)$$

If one expresses the e-functions backwards again by φ_0 , $\dot{\phi}_0/\omega$, φ_1 , and $\dot{\phi}_1/\omega$, one obtains

$$U_v = c' \left[D \left(\varphi_1 \frac{\dot{\phi}_1}{\omega} - \varphi_0 \frac{\dot{\phi}_0}{\omega} \right) + 2c^2 (1 - D^2) (1 - e^{-2\delta t_1}) \mp (\text{sgn } F) \frac{b}{\omega^2} D \left(\frac{\dot{\phi}_0}{\omega} - \frac{\dot{\phi}_1}{\omega} \right) \right] \quad (6.5)$$

and, accordingly, for the difference of the two energy values

$$\begin{aligned}\Delta U = U_V - U_Z &= \int_0^{t_1} (b'\dot{\phi})\dot{\phi} dt - \int_0^{t_1} m\dot{\phi} dt \\ &= c' \left\{ D \left(\varphi_1 \frac{\dot{\phi}_1}{\omega} - \varphi_0 \frac{\dot{\phi}_0}{\omega} \right) + 2c^2(1 - D^2) \left(1 - e^{-2\delta t_1} \right) \right. \\ &\quad \left. + (\operatorname{sgn} F) \frac{b}{\omega^2} \left[D \left(\frac{\dot{\phi}_0}{\omega} - \frac{\dot{\phi}_1}{\omega} \right) + (\varphi_0 - \varphi_1) \right] \right\} \quad (6.6)\end{aligned}$$

First, it shall be shown that ΔU vanishes for reversal points belonging to periodic motions.

The periodic motions themselves may be found without an energy consideration: For periodic motions there exist between the reversal values $\varphi_0, \dot{\phi}_0/\omega$ at the beginning of the interval and the reversal values $\varphi_1, \dot{\phi}_1/\omega$ at the end of it the relations

$$\varphi_0 = -\varphi_1 \quad (6.7a)$$

and

$$\frac{\dot{\phi}_0}{\omega} = -\frac{\dot{\phi}_1}{\omega} \quad (6.7b)$$

The period then is the double interval-length $T = 2t_1$. With the aid of the equations (2.13a) and (2.13b), (6.7a) and (6.7b) may be put in the form

$$2C \cos \epsilon + (\operatorname{sgn} F) \frac{b}{\omega^2} = -2C \cos(\epsilon + \nu t_1) e^{-\delta t_1} + (\operatorname{sgn} F) \frac{b}{\omega^2} \quad (6.8a)$$

and

$$\cos(\epsilon + \sigma) = -\cos(\epsilon + \sigma + \nu t_1) e^{-\delta t_1} \quad (6.8b)$$

The equations (6.8a) and (6.8b) represent two relations between the integration constants C and ϵ and the quantity σ characteristic of the system on one hand and the interval length vt_1 on the other. If one expresses the angles ϵ and σ with the aid of (2.11) and (2.5) by the reversal values ϕ_0 and $\dot{\phi}_0/\omega$, one achieves (after a longer calculation) the equations

$$\phi_0 = \mp (\text{sgn } F) \frac{b}{\omega^2} \frac{\left(\frac{D}{\sqrt{1-D^2}}\right) \sin(\sqrt{1-D^2} \omega t_1) - \sin D \omega t_1}{\cos(\sqrt{1-D^2} \omega t_1) + \cos D \omega t_1} \quad (6.9a)$$

$$\frac{\dot{\phi}_0}{\omega} = \pm (\text{sgn } F) \frac{b}{\omega^2} \frac{1}{\sqrt{1-D^2}} \frac{\sin(\sqrt{1-D^2} \omega t_1)}{\cos(\sqrt{1-D^2} \omega t_1) + \cos D \omega t_1} \quad (6.9b)$$

as parametric representation of the "curve of the periodic solutions" in the plane of reversal values.

This curve is a spiral-like curve (cf. figures 28a and 28b); the interval length t_1 is a parameter. The curve starts for $t_1 = 0$ at the origin of the coordinate system and runs for $t_1 \rightarrow \infty$ toward the point $\phi_0 = \pm 1$, $\dot{\phi}_0/\omega = 0$. For $vt_1 = \pi$ the curve intersects the ordinate axis. The points $vt_1 < \pi$ belong (according to the aforesaid) to the system A, the points $vt_1 > \pi$ to the system B. Moreover, the curve ends where vt_1 becomes $v\bar{t}_1$. It ends, therefore, on the "large limiting ellipse" (which limits the region of the rest points); it can be shown that that point is simultaneously the point of intersection with the small limiting ellipse.

If one now substitutes in (6.6) — after having expressed C also by ϕ_0 and $\dot{\phi}_0/\omega$ — the reversal values according to (6.9) and takes into consideration that moreover $\phi_1 = -\phi_0$ and $\dot{\phi}_1/\omega = -\dot{\phi}_0/\omega$, one finds

$$\Delta U = 0$$

or

$$U_V = U_Z \quad (6.10)$$

As before, the value $D = 0.1$ of the amount of damping is taken as a basis also for the curves of figures 27 and 28. For other values of the amount of damping the curves of the periodic motions appear as indicated in figure 18 which refers to reversal points a. For $D = 0$ the curve degenerates into a straight line which coincides with the φ_0/ω - axis.

Furthermore, the sign assumed by the energy difference ΔU outside and inside of the curve of the periodic motions (6.9) is of interest.

A generally valid discussion of the sign with the aid of the equation (6.6) seems difficult. However, a large number of numerical examples show that for system A in first and third quadrants, for system B in the sector between the φ_0 - axis and the ray OSQ (fig. 28) the motions always converge toward the periodic ones, from reversal points outside of the curve of periodic motions with decreasing deflections and from reversal points inside of that curve with increasing deflections. In the sector between the rays OSQ and OR the motions go from "inside" points with increasing deflections toward the periodic ones.

The arc RS of the curve of the periodic motions is "unstable", that is, from reversal points in its vicinity the motion goes, in case the point lies "inside", toward a periodic motion on the arc section lying farther to the outside, in case the point lies "outside", the motion is damped in the customary manner (leading to a rest point).

A good survey of the course of the motion discussed just now and therewith of the problem of the removal or addition of energy may be obtained from figures 19 to 24. Figures 20, 21, and 24 show, against the non-dimensional interval length νt_1 , the pertinent value of reversal φ_0 at the beginning and φ_1 at the end of the interval plotted for various values ρ . If one selects a first reversal point, for instance in figure 21, all the following are easily determined by drawing in a stepped line. Starting from the reversal point S_4 , one no longer obtains an intersection point with the curve $|\varphi_0|$. The point S_4 corresponds to a point in the domain of the small limiting ellipse. For $\rho = 0$ one must plot $\ddot{\varphi}_0/\omega$ since all $\varphi_n = 0$, ($n = 1, 2, 3, \dots$), see figure 19. For $\rho \rightarrow \pm\infty$, νt_1 always

equals π and, moreover, $\dot{\phi}_0$ always equals zero. Thus one has to select here also another manner of plotting. We selected the one used by K. Bögel⁶ since for $\rho \rightarrow \infty$ our problem transforms into the one treated by Bögel. For this case a simple relation between the reversal values ϕ_0 and ϕ_1 may be given (cf. figs. 22 and 23).

A further problem arises in connection with the energy considerations. Even the free oscillation of a damped system causes removal of energy. If the control of the system is to damp the motion more rapidly, the decrease of the maximum deflections must be stronger than for the free oscillation. For the free oscillation the ratio of two successive maximum deflections is $e^{-\frac{Dv}{v} \pi}$. For the controlled oscillation the ratio is not constant, and the time t_M between two maximum deflections is not constant, either. It is, however, assumed that t_M always lies between the correlated interval t_{1I} and t_{1II} (cf. fig. 25a). This relation may be proved with the aid of the phase diagram (fig. 25b). For this purpose the construction is not continued from the reversal point S_2 in the customary manner, but S_2 is - since we are interested only in the absolute values of the maximum ϕ - values - mirrored at the zero point; then one may continue to operate in the second interval with the same point of convergence, as can be understood immediately. One has $\angle S_0 P S_1 = vt_{1I}$; $\angle S_1' P S_2' = vt_{1II}$; $\angle S_1 P S_1' = vt_M$. By construction,

$$\angle S_1' P S_2' < \angle S_1 P S_1' < \angle S_0 P S_1$$

By the construction a simple correlation between t_M and the adjoining interval lengths is given. For this reason there is for system A $vt_M \leq \pi$ and for system B $\pi \leq vt_M \leq vt_{1I}$ (cf. p. 27).

⁶Bögel, K.: Das Verhalten gedämpfter und aufschaukelnder freier Schwinger unter der gleichzeitigen Einwirkung einer konstanten Reibungskraft. (The behavior of damped and amplitude-increasing free oscillators under simultaneous influence of a constant frictional force.) Ing.-Archiv 12, p. 247.

One may now plot the ratio of two successive maximum values (construction cf. fig. 25b) against $v t_M$ (fig. 26). The uncontrolled oscillation is represented in this diagram by the point κ ($v t_M = \pi$).

In order to be enabled to compare the damping of the motion in the controlled and in the uncontrolled system, one has to take into consideration that in the controlled system the change of the maximum deflection occurs at different times. The rate of damping in the controlled system is (φ_{\max_n} of the uncontrolled motion was denoted

by φ_{Un} , φ_{\max_n} of the controlled motion by φ_{Gn}):

$$v_G = \frac{|\varphi_{Gn+1}| - |\varphi_{Gn}|}{v t_M}$$

In the uncontrolled system, that is, for a free damped oscillation

$$v_U = \frac{|\varphi_{Un+1}| - |\varphi_{Un}|}{\pi}$$

Only when the ratio v_G/v_U exceeds 1 does the control fulfill its purpose. Thus one forms the ratio v_G/v_U for equal initial value $\varphi_{G1} = \varphi_{U1}$:

$$\frac{v_G}{v_U} = \frac{\frac{|\varphi_{Gn+1}|}{|\varphi_{Gn}|} - 1}{\frac{|\varphi_{Un+1}|}{|\varphi_{Un}|} - 1} \frac{\pi}{v t_M} = \frac{\frac{|\varphi_{Gn+1}|}{|\varphi_{Gn}|} - 1}{\left(e^{-\frac{D}{\sqrt{1-D^2}}\pi} - 1\right) \frac{v t_M}{\pi}}$$

The numerator $\frac{|\varphi_{Gn+1}|}{|\varphi_{Gn}|} - 1$ can immediately be taken from figure 26⁷.

The denominator $\left(e^{-\frac{D}{\sqrt{1-D^2}} \pi} - 1 \right) \frac{v t_M}{\pi}$ represents a straight line which is drawn in for comparison. It goes through the zero point and through the point K. For all points lying below the drawn-in straight line f, the ratio is $v_G/v_U > 1$, that is, the controlled motion is damped more quickly than the uncontrolled one. This, however, signifies that, as a rule, the system A will be preferable. We shall return to this problem in section 7.

7. Comprehensive Discussion of the Possible Courses of Motion (Discussion of Figures 27 and 28)

The perceptions gained in the previous paragraphs are summarized in figures 27 and 28. In both figures the type of line used indicates the type of the reversal points: Solid curves correspond to the reversal points a, dashed ones to the reversal points b. Figures 27a and 27b show for system A and for system B, respectively, the curves of constant interval length t_1 (isochrones). Here one recognizes the fact mentioned in section 5, that in the system A the parameter is $t_1 \leq \pi/v$, in the system B, in contrast, $t_1 \geq \pi/v$. To the family

⁷Regarding the limit rest in figure 26 the following remark must be made: On p. 35 an upper limit is determined for the time interval $v t_M$ between the successive maximum values. Only maximum values within the sphere of normal operation of the control are considered, for instance φ_{G1} and φ_{G2} in the sketch, figure 26,

upper right. If, however, in the system B rest occurs in the course of a motion, one is inclined to select (in considering the damping of the controlled system) as the last pair of maximum values φ_{G2}

and φ_{U1} . The time interval $(v t_M)_R$ between these maximum values, however, may be larger than $v \bar{t}_1$, as can be easily taken from the phase curve: $\pi < (v t_M)_R < 2\pi$.

of these curves of constant interval length also belong those for $t_1 = 0$ and $t_1 = \bar{t}_1$, which had been denoted as the "small" and "large" limiting ellipses in sections 4 and 5, and the properties of which had been discussed there. In figure 29 the small and the large limiting ellipse are (for the system B and reversal points a) drawn separately once more and the significance of the individual domains resulting from section 4 is indicated.

Besides the small limiting ellipses which enclose the domain of the limit points, those domains are marked in the two figures 28a and 28b, whence one arrives at an end point after one or after two intervals. The domain whence a single interval leads to an end point is subdivided once more and it is noted whether this end point lies on this or the other side of the zero point.

For system B, aside from end points, rest points of the motion are also possible. Reversal points in the crescent-shaped domains lead after one or after two steps to such rest points.

After all that has been said so far, the course of the motion adjoining any reversal point is now clear and thus the time has come to decide on what parameters are useful for a desired course for given initial values. In section 1 it was shown that, when the actual time t is used as independent variable, the equation of motion contains the parameters D , ω , b , and ρ . Let us assume that D and ω are prescribed, that is, that we are dealing with an oscillating system the damping and natural frequency of which are given. Let the damping be insufficient and the requirement be made that the damping be improved by application of an intermittent control. It is, therefore, desired to attain in the shortest possible time, for prescribed initial values ϕ_a and $\dot{\phi}_a$, negligibly small

values ϕ and $\dot{\phi}$. We may now select b/ω^2 and ρ and decide, moreover, on control system A or B. If, for instance, a large b/ω^2 is selected, it may happen that almost all reversal values pertaining to our maximum possible initial values fall into the domain of the small limiting ellipse. That would mean that on the basis of the results of this report ("ideal" flip-flop control without lag) the motion becomes undefined after a short time or increases toward a periodic motion (cf. fig. 14 and p. 34). If, on the other hand, a small b/ω^2 is selected, the reversal values in question will lie within a domain of their plane which is so large that the limiting ellipses and the two curves of the periodic motions lie entirely within it. If now for instance a negative ρ is selected, we would have in system A for a large domain on initial reversal values an increase of amplitude; for

positive ρ , on the other hand, in general, damping extension into the domain of the small limiting ellipse would occur. Moreover it would have to be investigated whether the rate of damping is sufficiently high. If system B is selected, there would occur, for negative ρ , decrease down to the rest ellipse; for positive ρ an increase of amplitude occurs for a large domain (within the curve of the periodic solutions). Since the ellipse of the rest points is always larger than that of the end points (small limiting ellipse), system B with negative ρ will certainly be more unfavorable than A with positive ρ . Let us, therefore, decide on system A and positive ρ . Now arises the problem what magnitude of ρ to select. For this purpose one must consider the curve of damping against the time t , that is, the figures 19 to 22 and figure 26. According to these figures, one is at first very much inclined to select a rather large ρ . However, the following fact is opposed to that: As we know, one enters rather soon the domain of the small limiting ellipse in which the motion, under the effect of a control without lag, is no longer defined. It has been pointed out before, on p.20, that one again obtains defined phenomena of motion in this domain, if one takes the lag into consideration (cf. the report by K. Scholz⁸ made in connection with our report). It is now shown that, with these facts taken into consideration, smaller values of ρ are preferable since they cause a faster damping of the motion within the small limiting ellipse⁹.

Translated by Mary H. Mahler
National Advisory Committee
for Aeronautics.

⁸Scholz, K.: Über die Bewegungen eines Systems von einem Freiheitsgrad unter dem Einfluss einer Schwarz-Weiss-Steuerung mit Schaltverschiebungen. (On the motions of a system of one degree of freedom under the effect of a flip-flop control with lag of reversals.)

⁹Prof. Fischel (DFS) called our attention to the fact that it is shown in a report by Golling (shortly to be published) that for very small time lags the motion turns into a creeping toward zero ($\varphi = e^{-t/\rho}$).

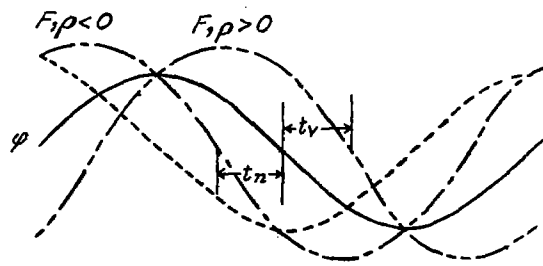


Figure 1.- Leading ($\rho > 0$) and lagging ($\rho < 0$) control functions.

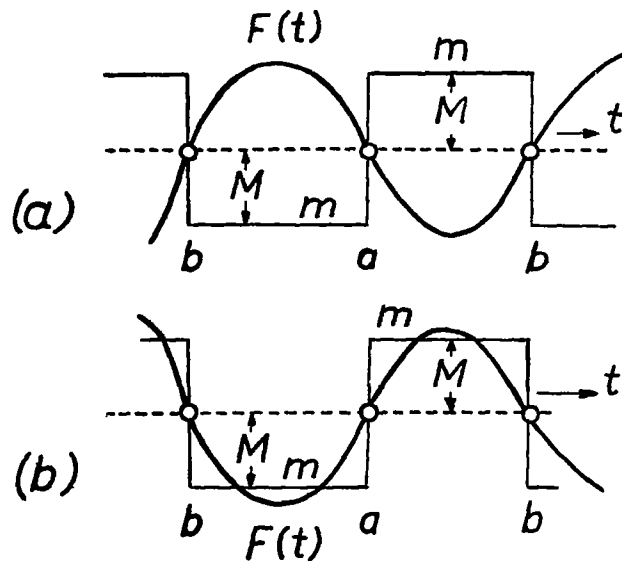


Figure 2.- Course of the control function F and of the control moment m for system A and system B.

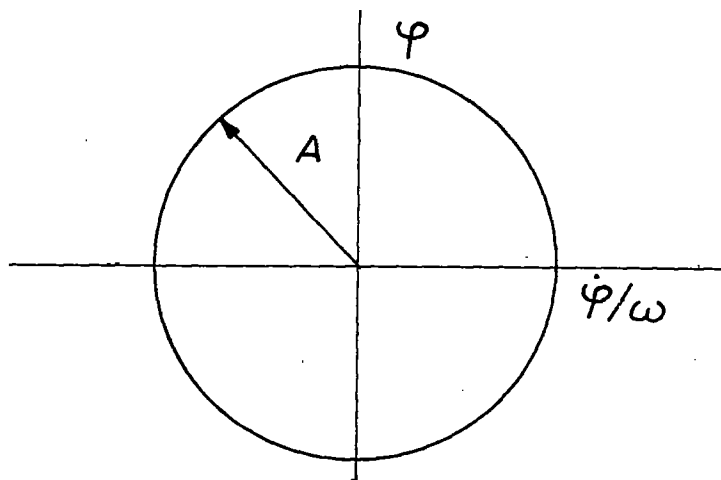


Figure 3.- Phase curve of a harmonic oscillation.

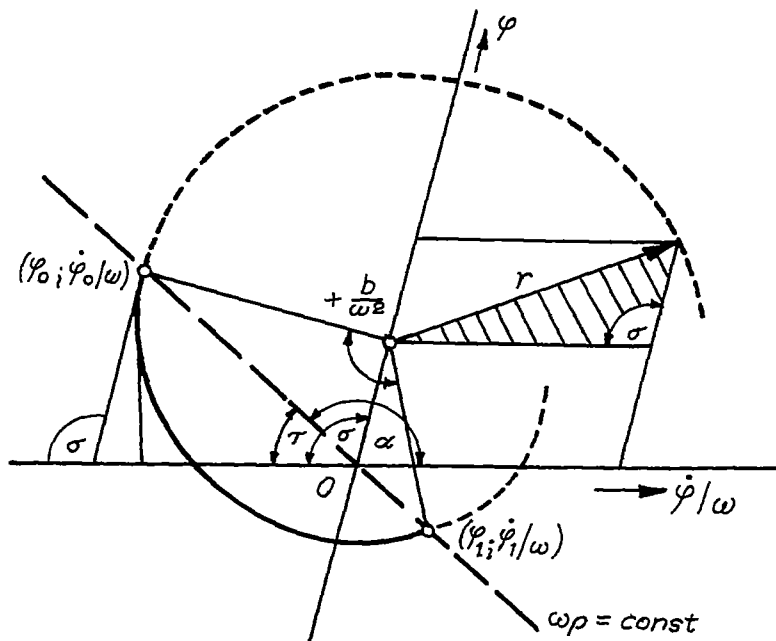


Figure 4.- Logarithmic spiral as phase curve in an oblique-angled coordinate system.

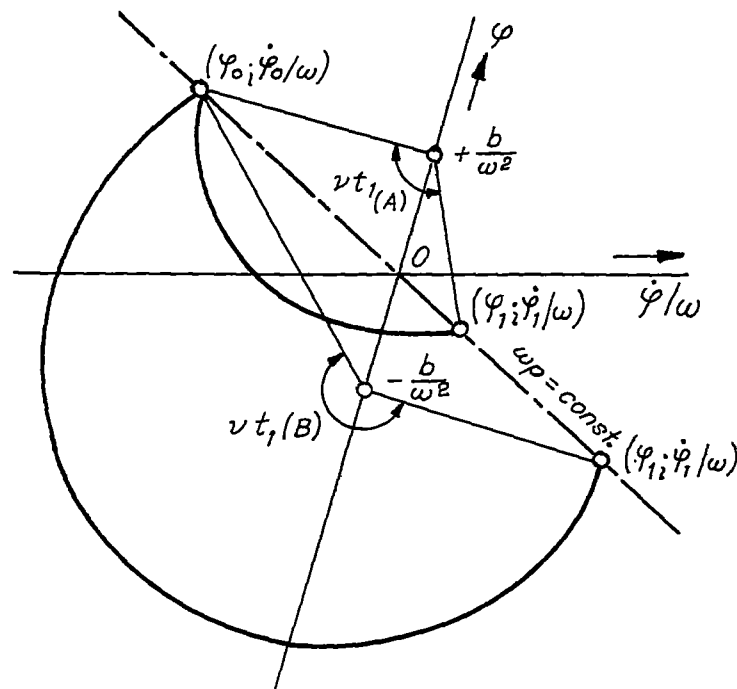


Figure 5.- Interval lengths for system A and system B.

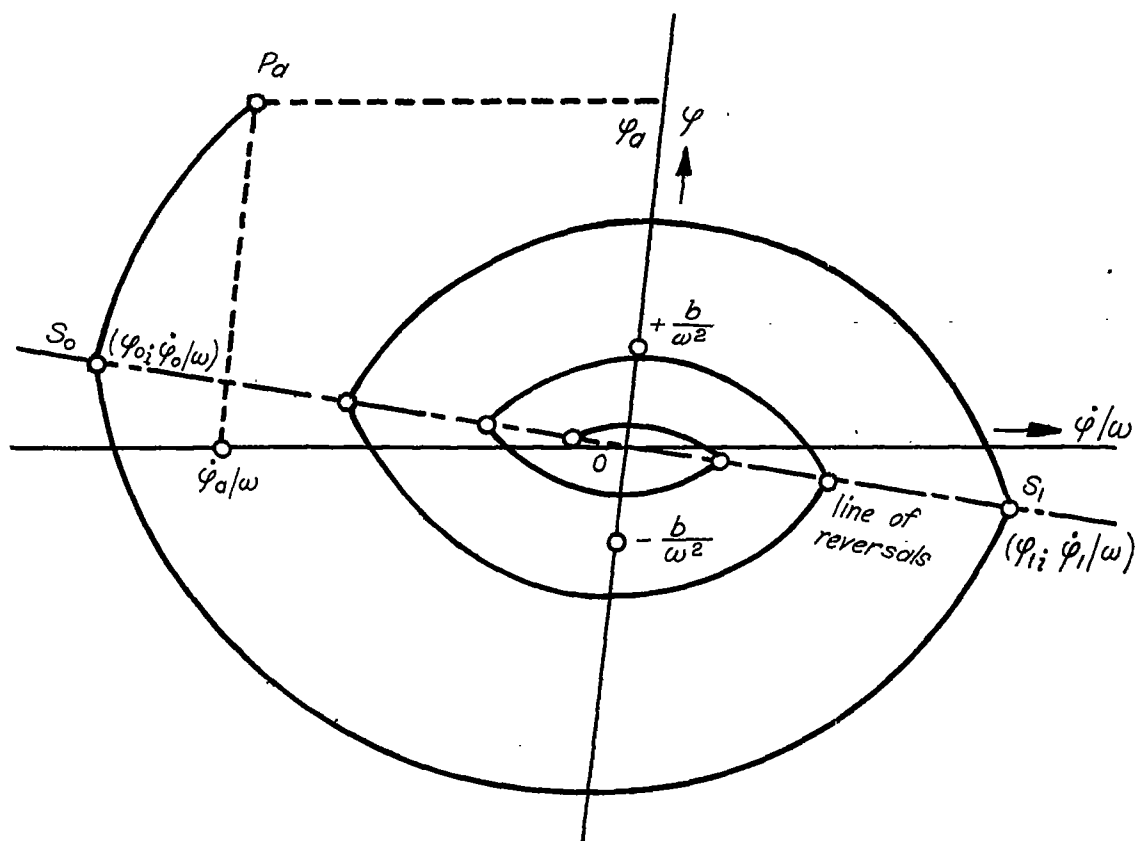


Figure 6.- Phase curve of a motion, composed of sections of logarithmic spirals.

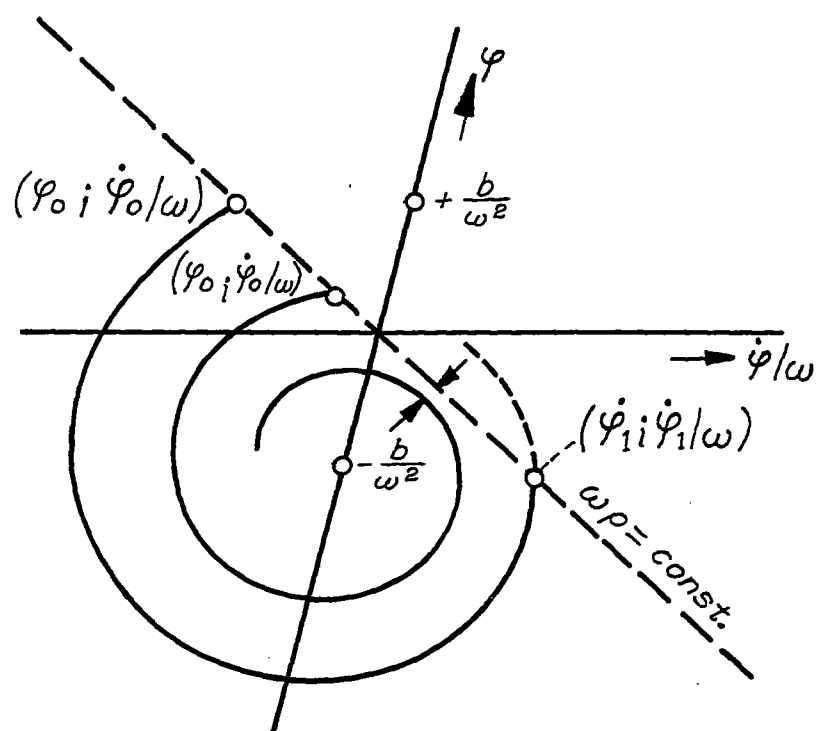


Figure 7.- Phase curve of a motion with control coming to "rest."

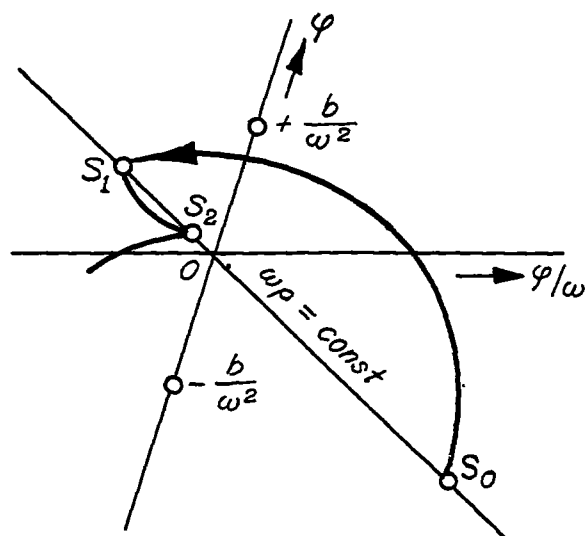


Figure 8.- End point S_2 of a motion.

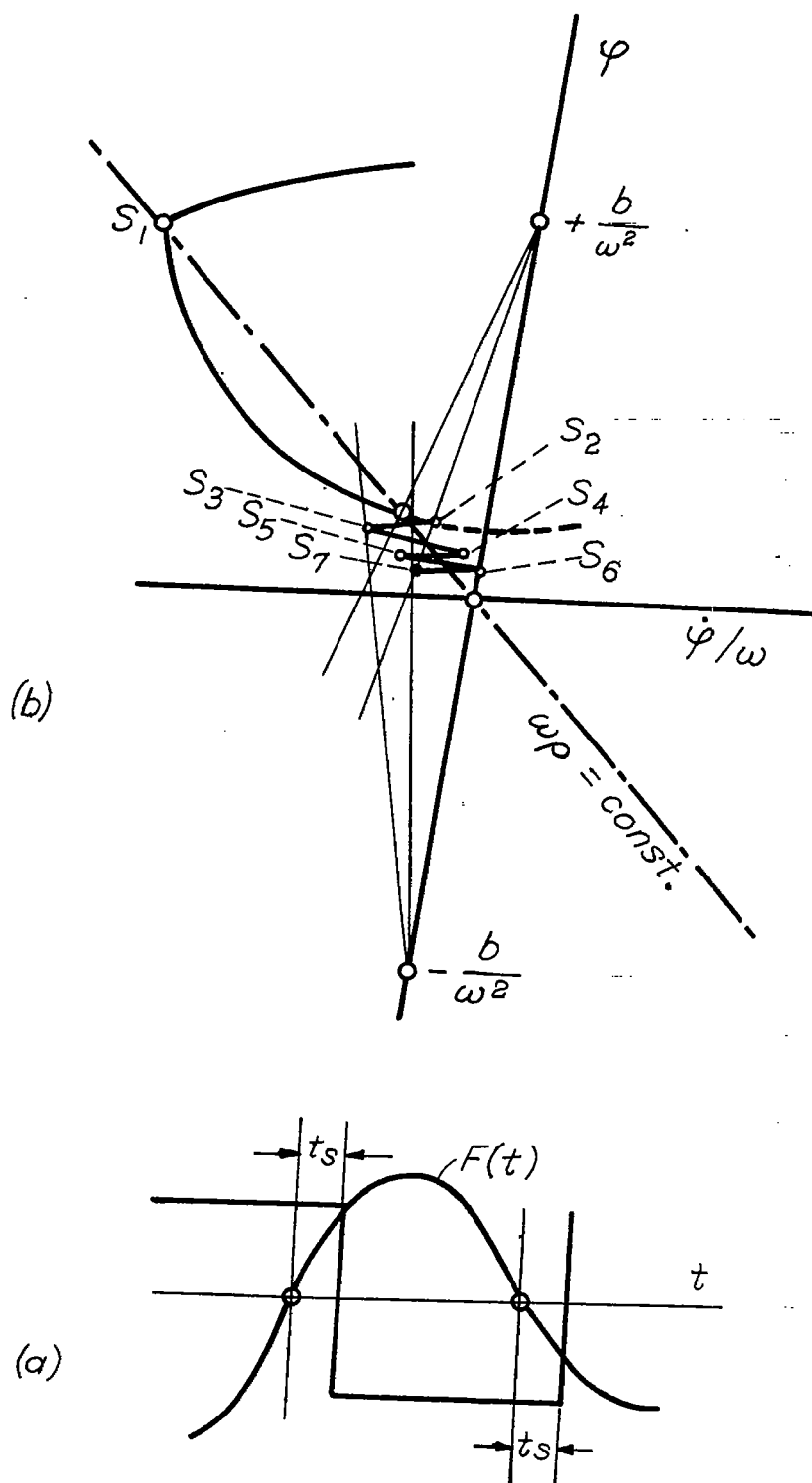


Figure 9.- Phase curve of a motion with lagging control surface (constant lag period $v\Delta t = 5^\circ$).

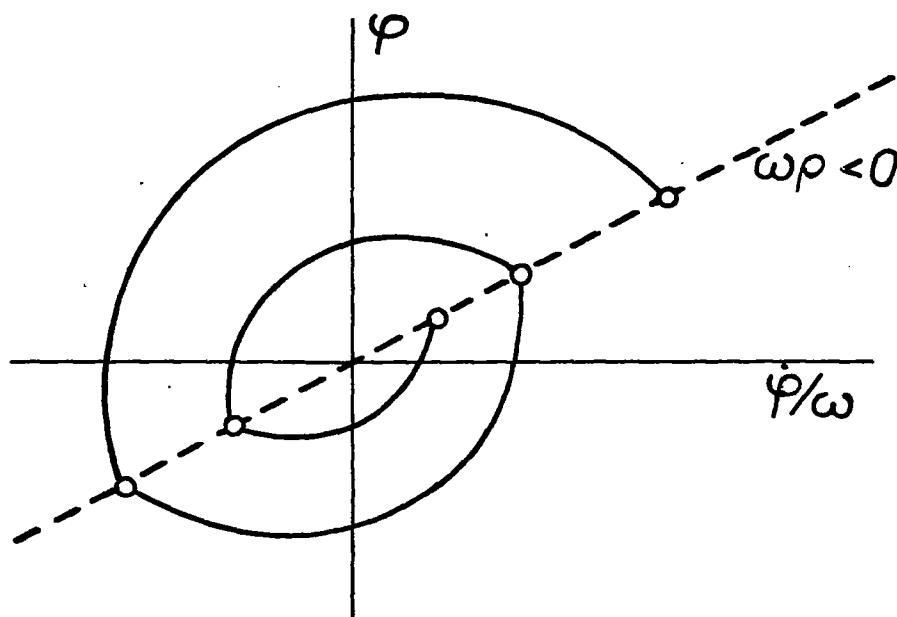


Figure 10.- Phase curve in Cartesian coordinates (schematic).

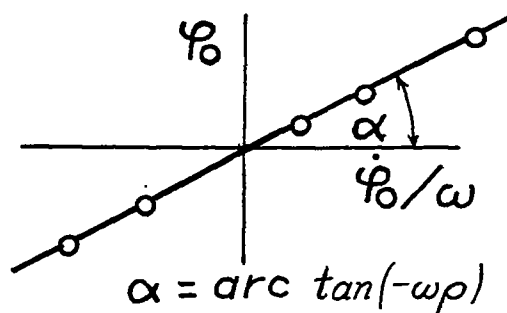


Figure 11.- Reversal points.

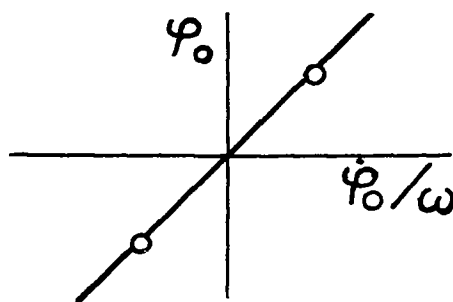


Figure 12.- Reversal points of a periodic motion.

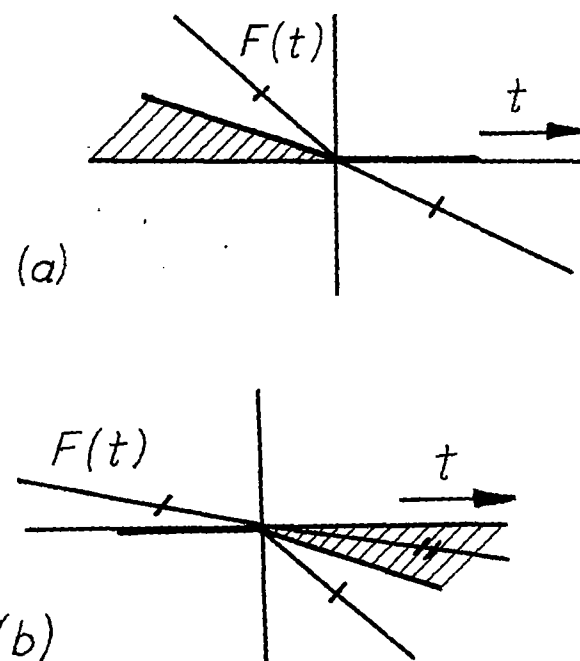


Figure 13.- "Breaking" of the control function.

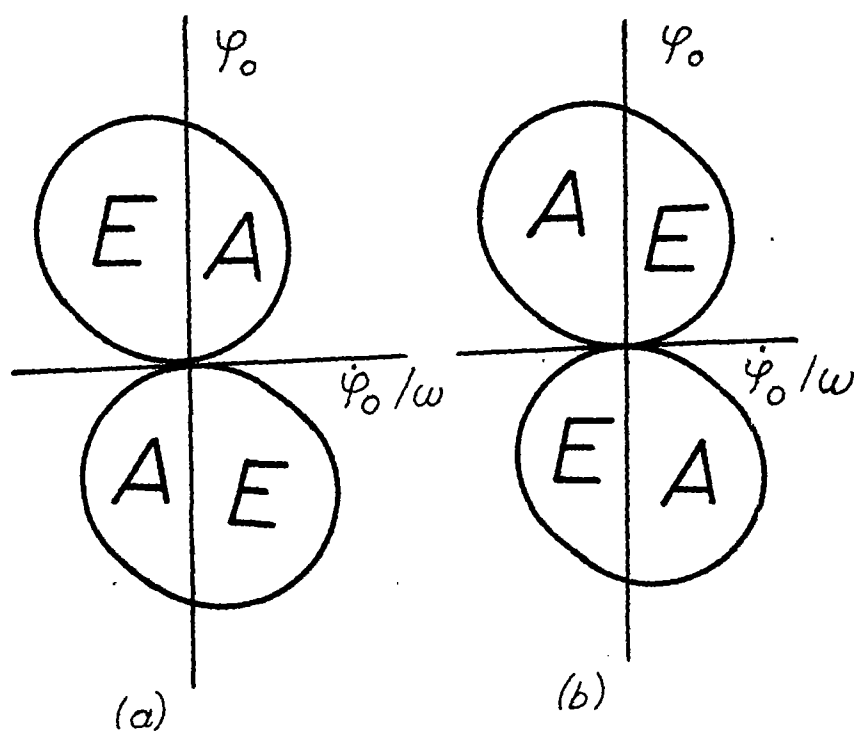


Figure 14.- Small limiting ellipses for system A and system B with initial and end values.

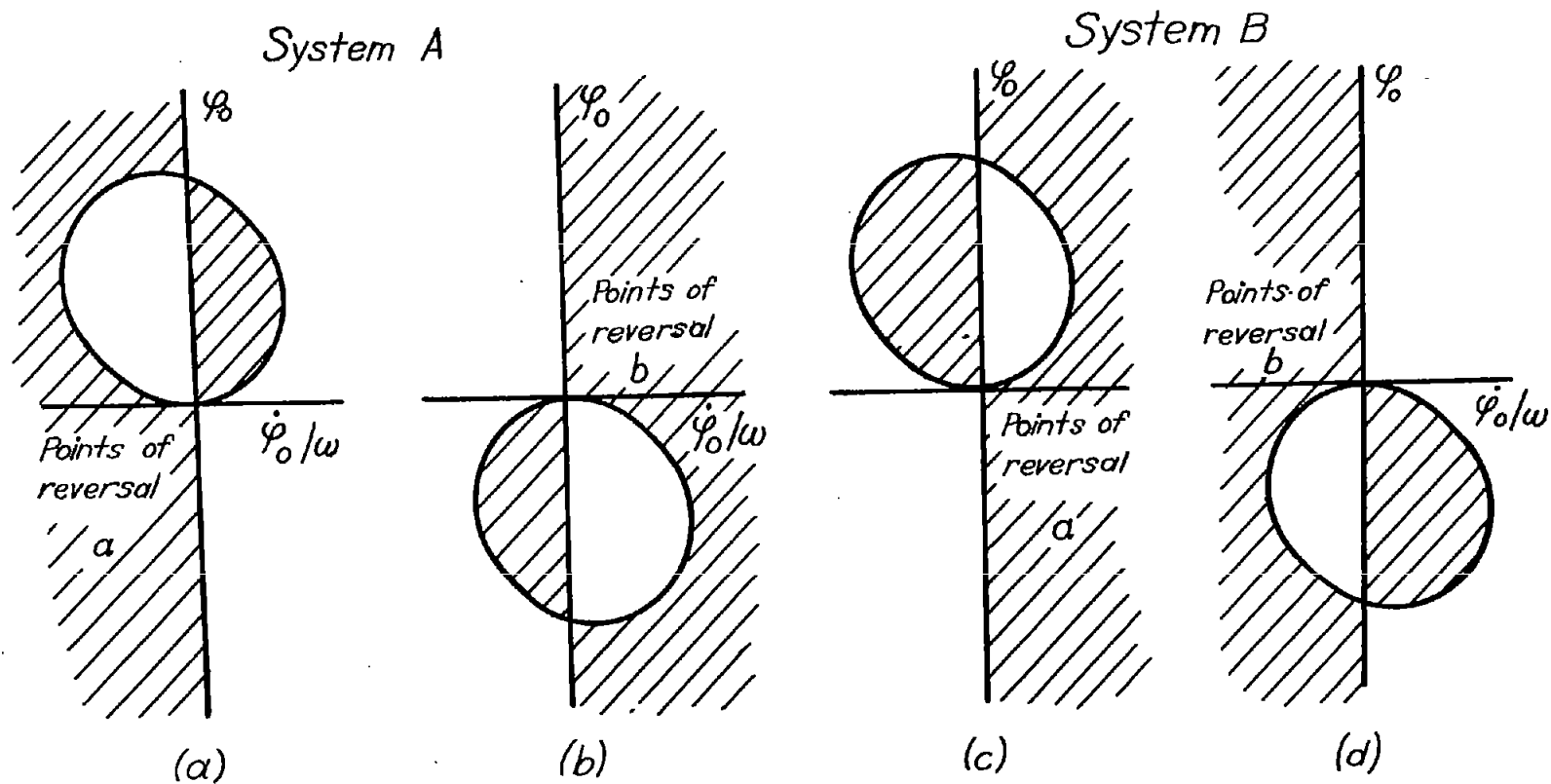


Figure 15.- Domains of the reversal points a and b.

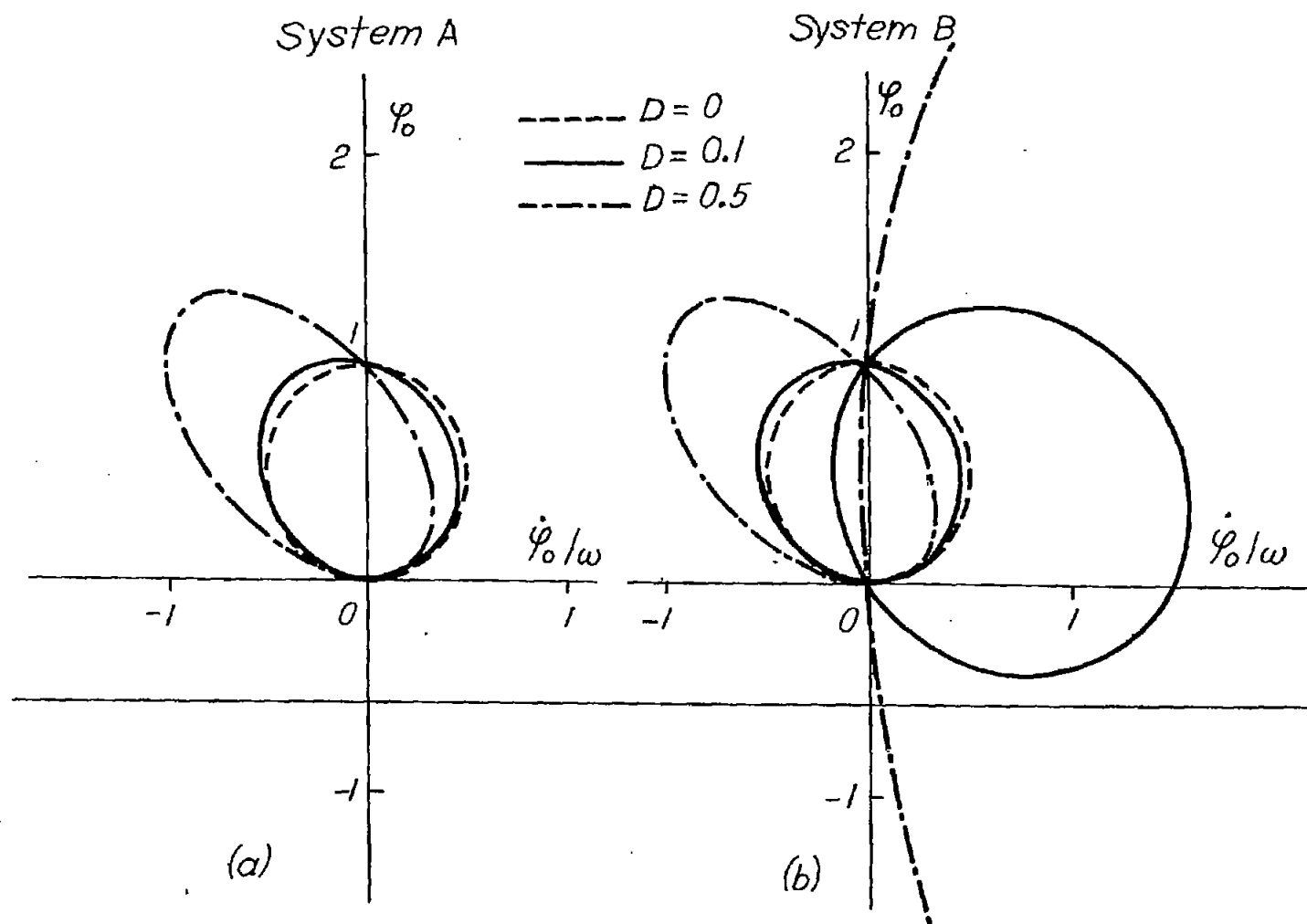


Figure 17.- Small and large limiting ellipses for various values D and $\frac{b}{\omega^2} = 1$.

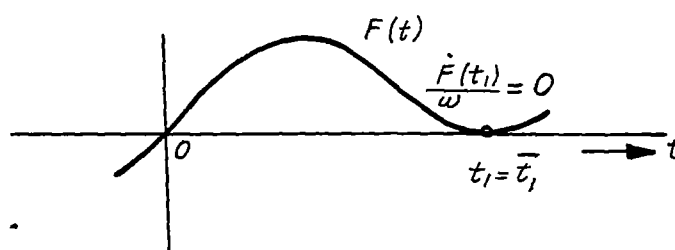
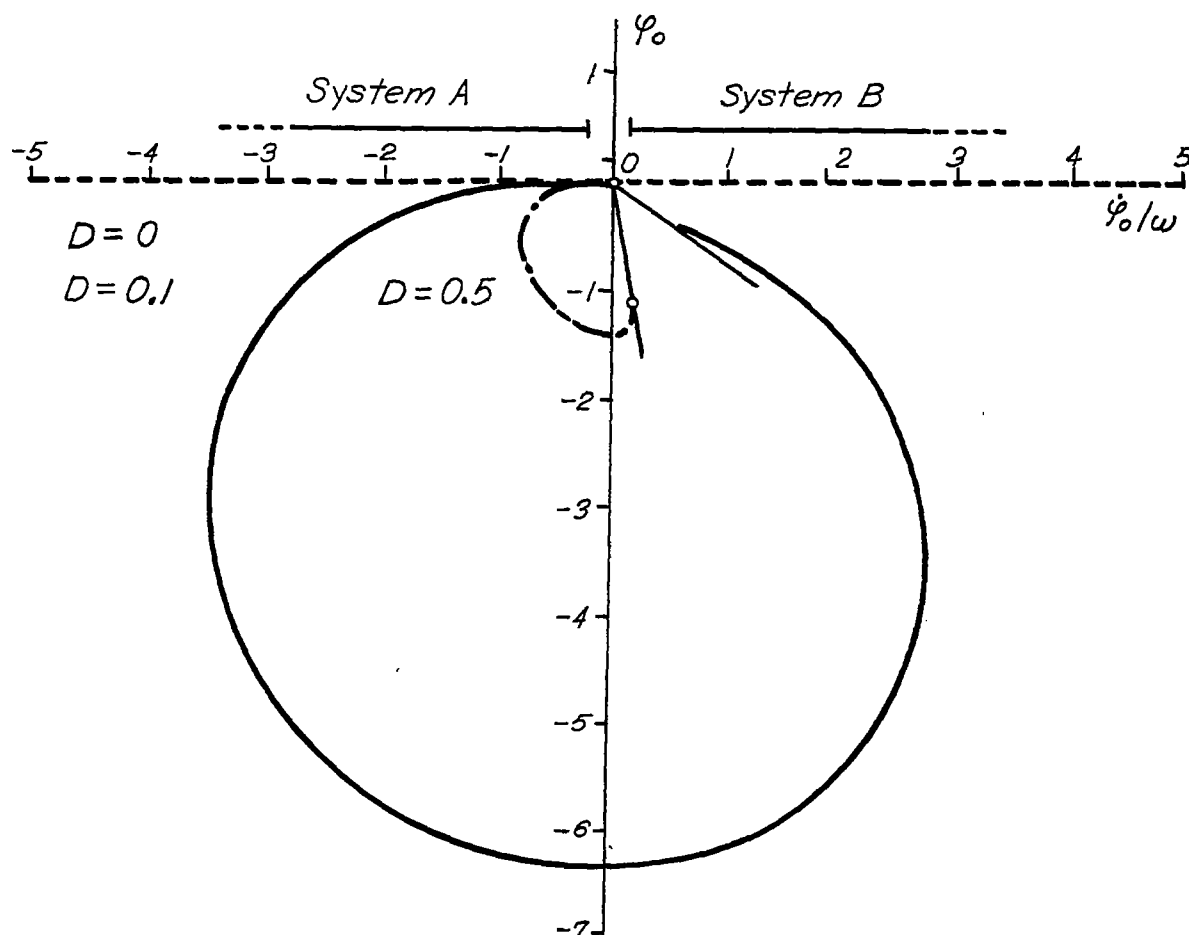


Figure 16.- Limiting "rest" condition.

Figure 18.- Reversal values a for periodic motions for various amounts of damping D ; $\frac{b}{\omega^2} = 1$.

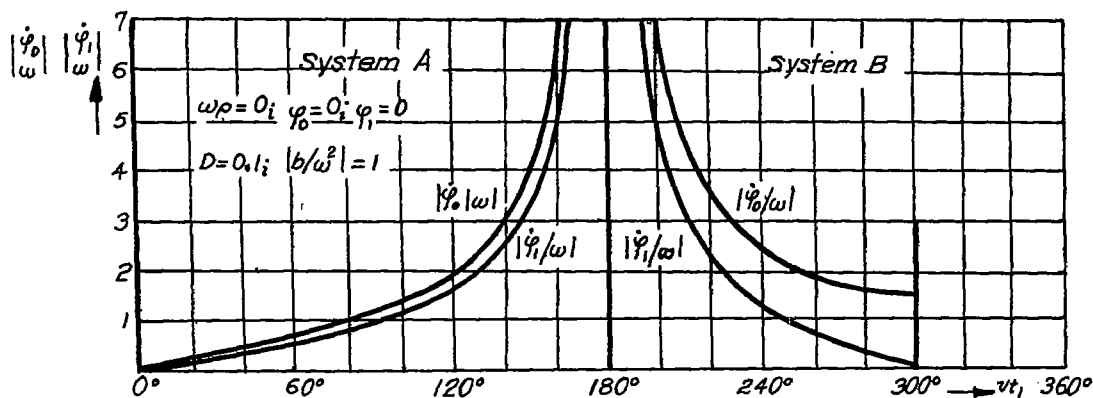


Figure 19.- Survey of the decrease and increase, respectively, of the reversal values of controlled motions.

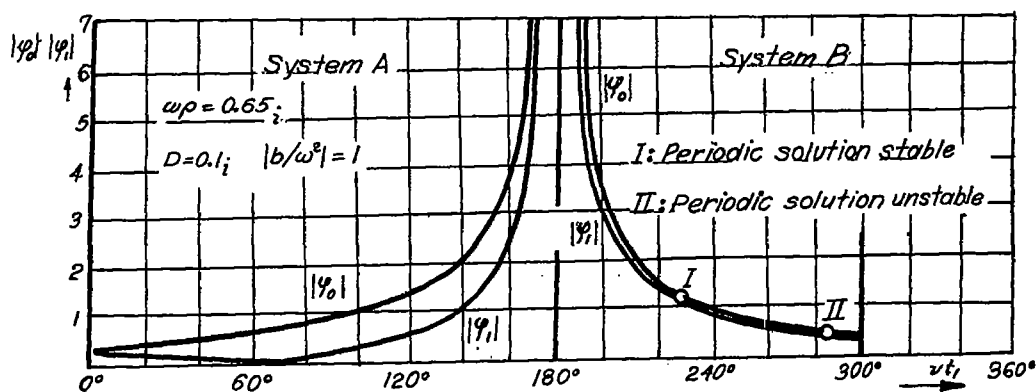


Figure 20.- Survey of the decrease and increase, respectively, of the reversal values of controlled motions.

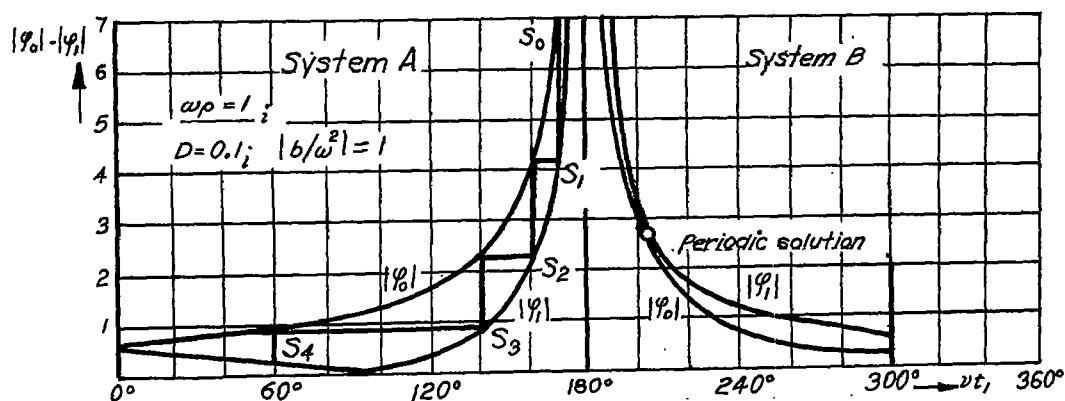


Figure 21.- Survey of the decrease and increase, respectively, of the reversal values of controlled motions.

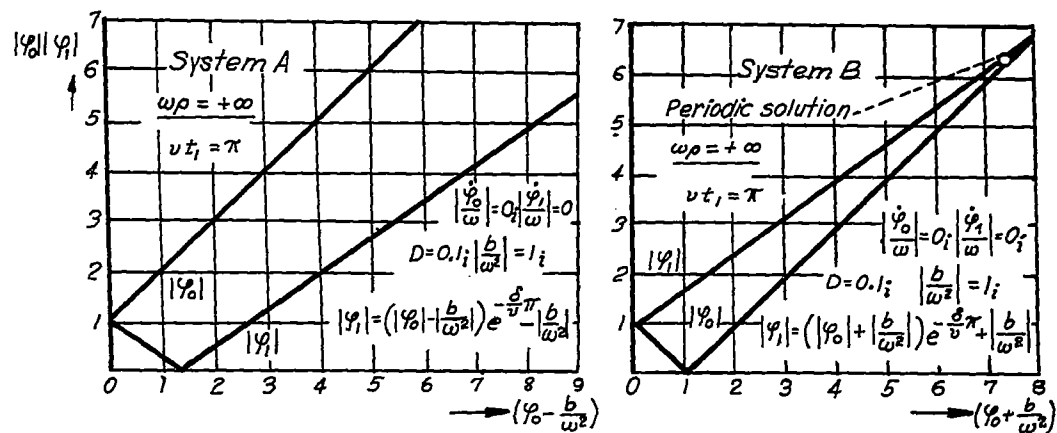


Figure 22.- Survey of the decrease and increase, respectively, of the reversal values of controlled motions.

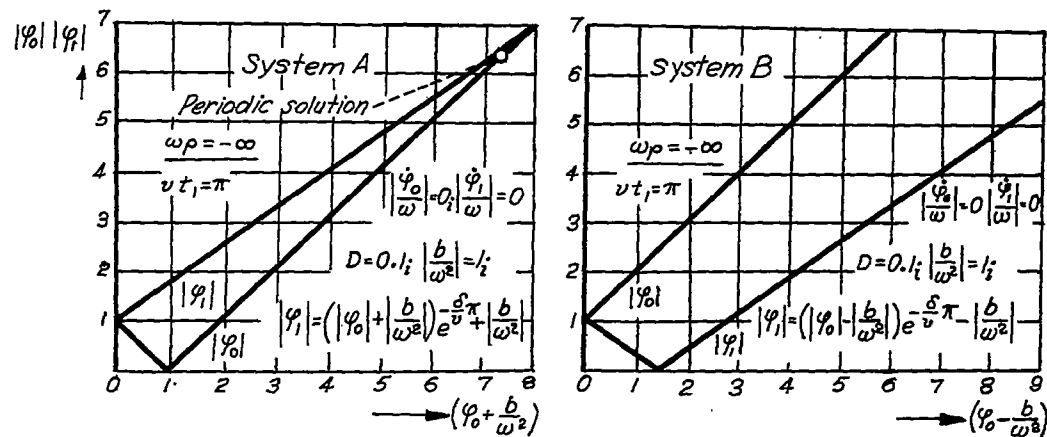


Figure 23.- Survey of the decrease and increase, respectively, of the reversal values of controlled motions.

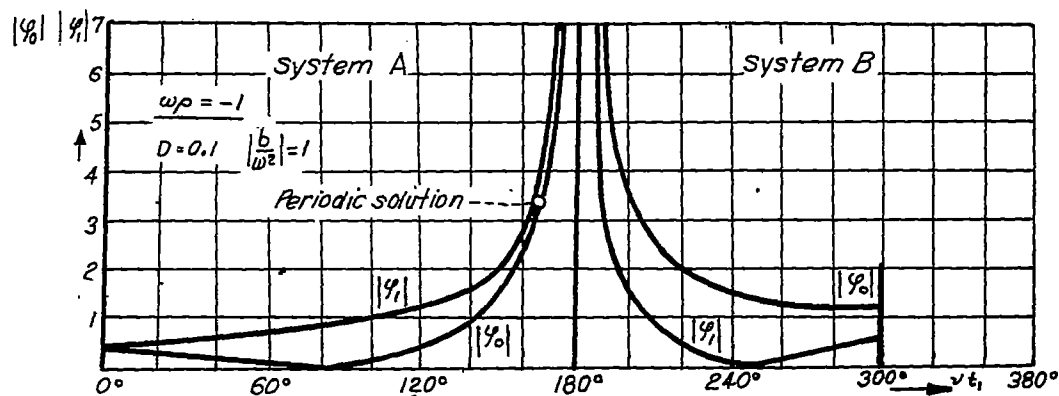


Figure 24.- Survey of the decrease and increase, respectively, of the reversal values of controlled motions.

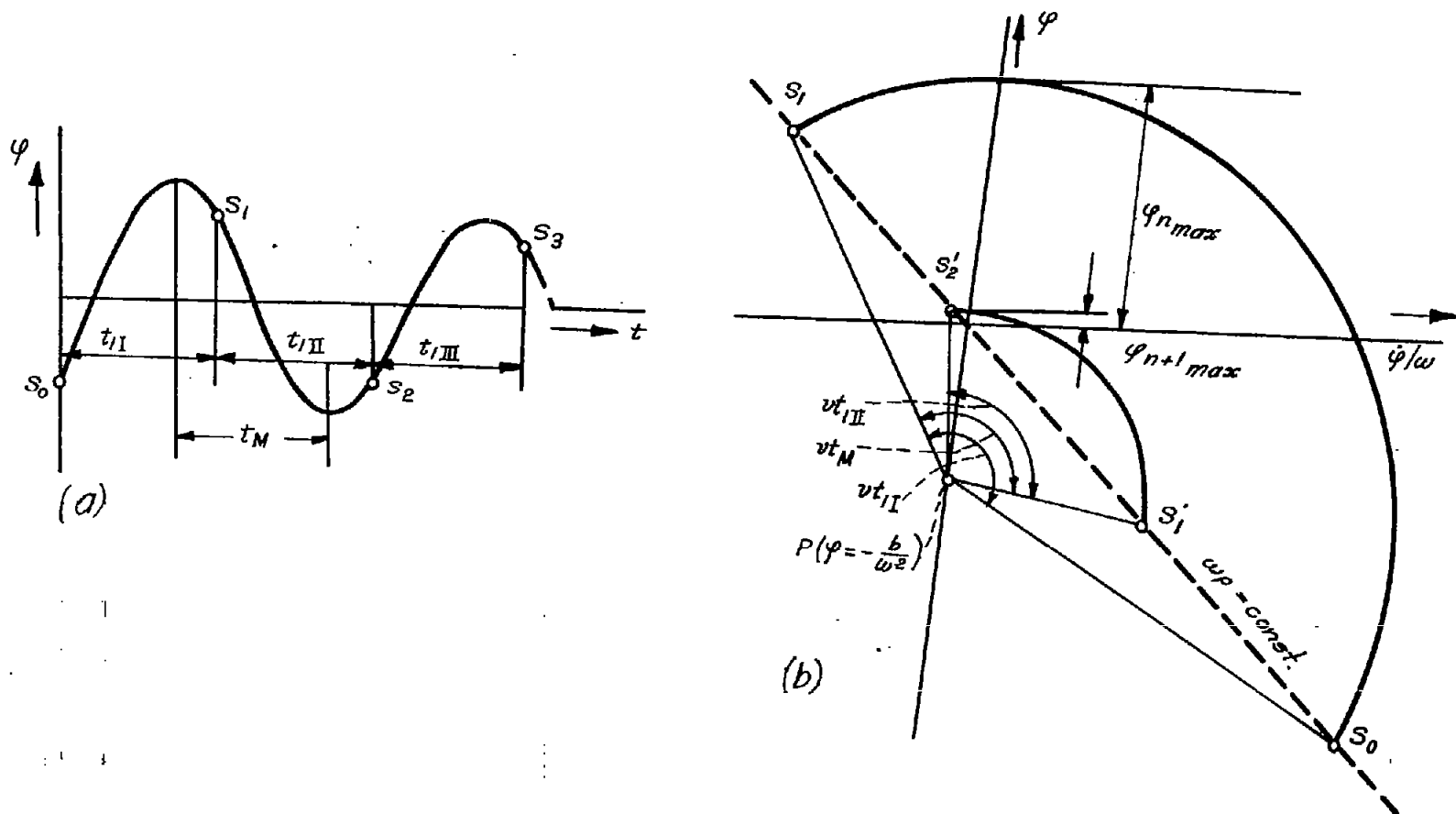


Figure 25.- Concerning the construction of t_M and ϕ_{\max_n} .

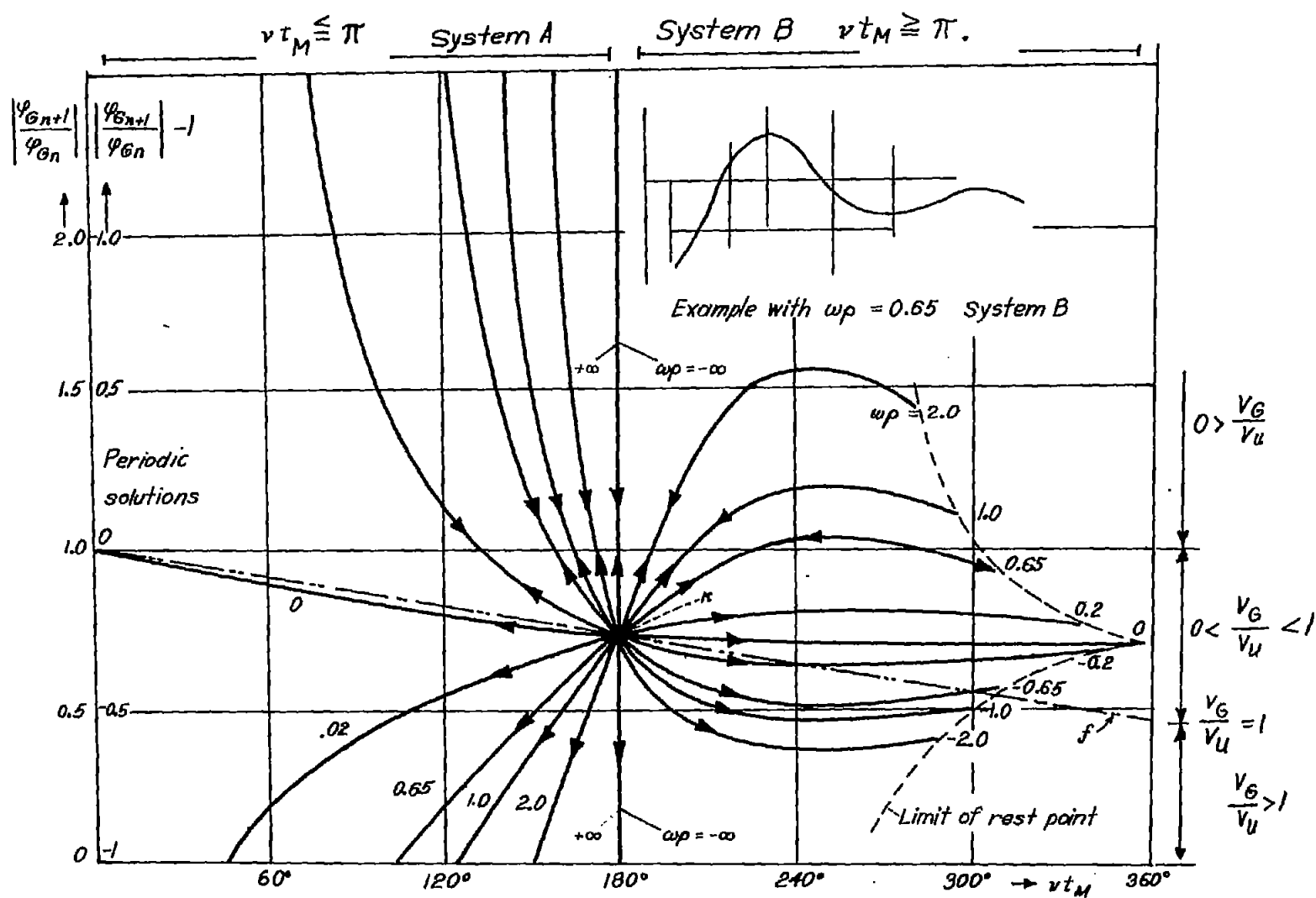


Figure 26.- Ratio of two successive maximum deflections plotted against the time interval between them.

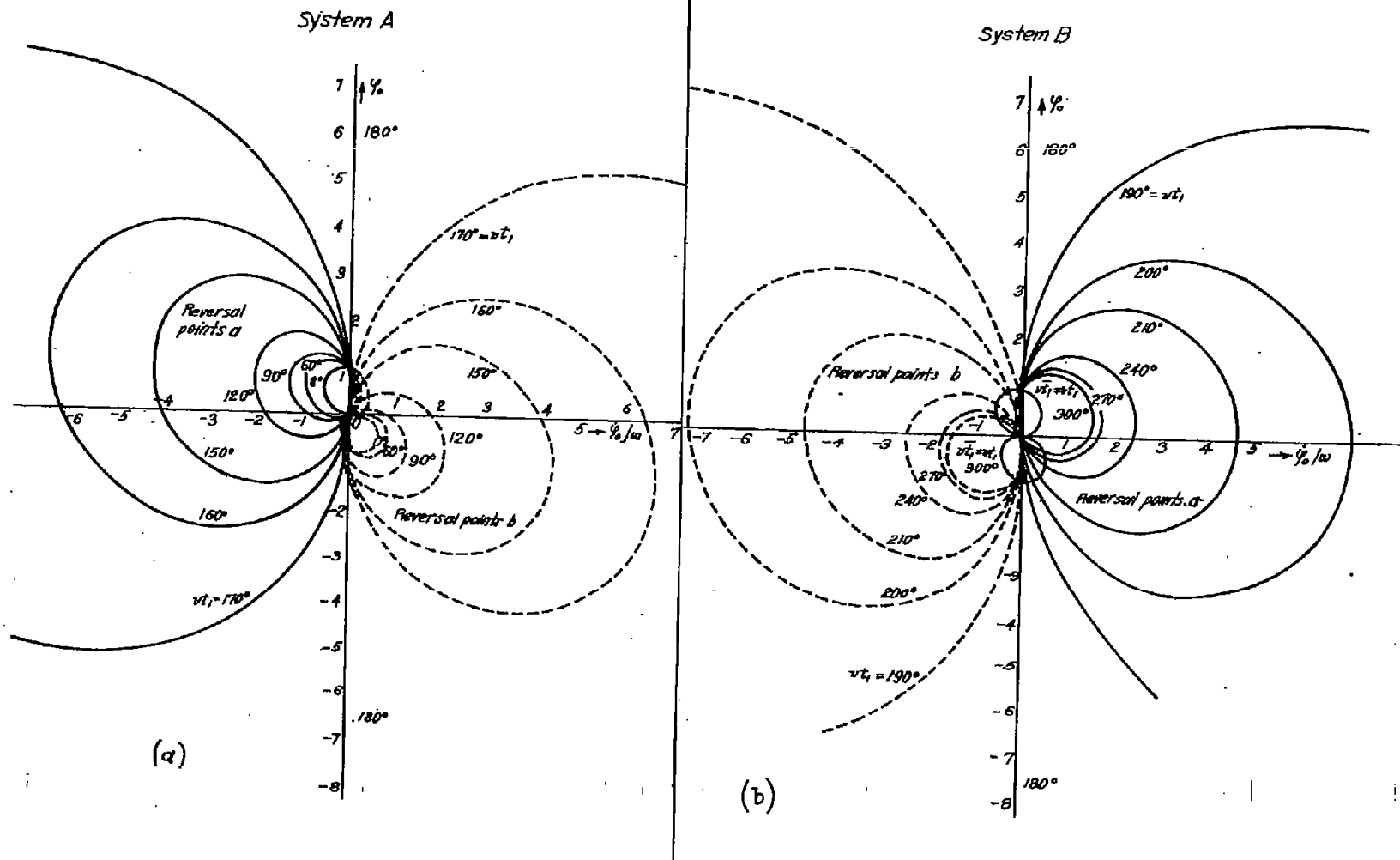


Figure 27.- Curves of constant interval length t_1 (isochrones) in the plane of the reversal values for system A and for system B.

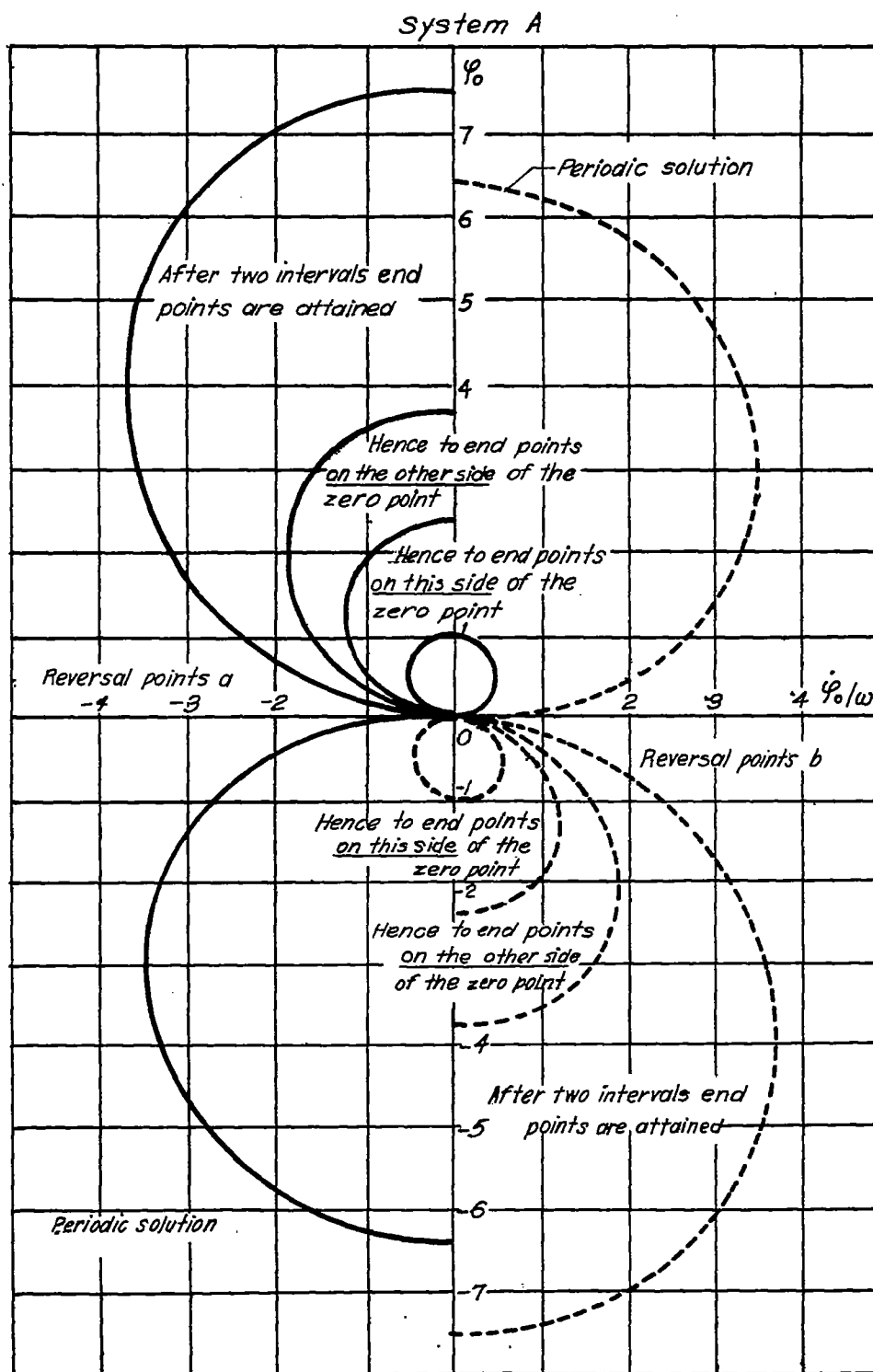


Figure 28(a).- Division of the plane of reversal values. System A.

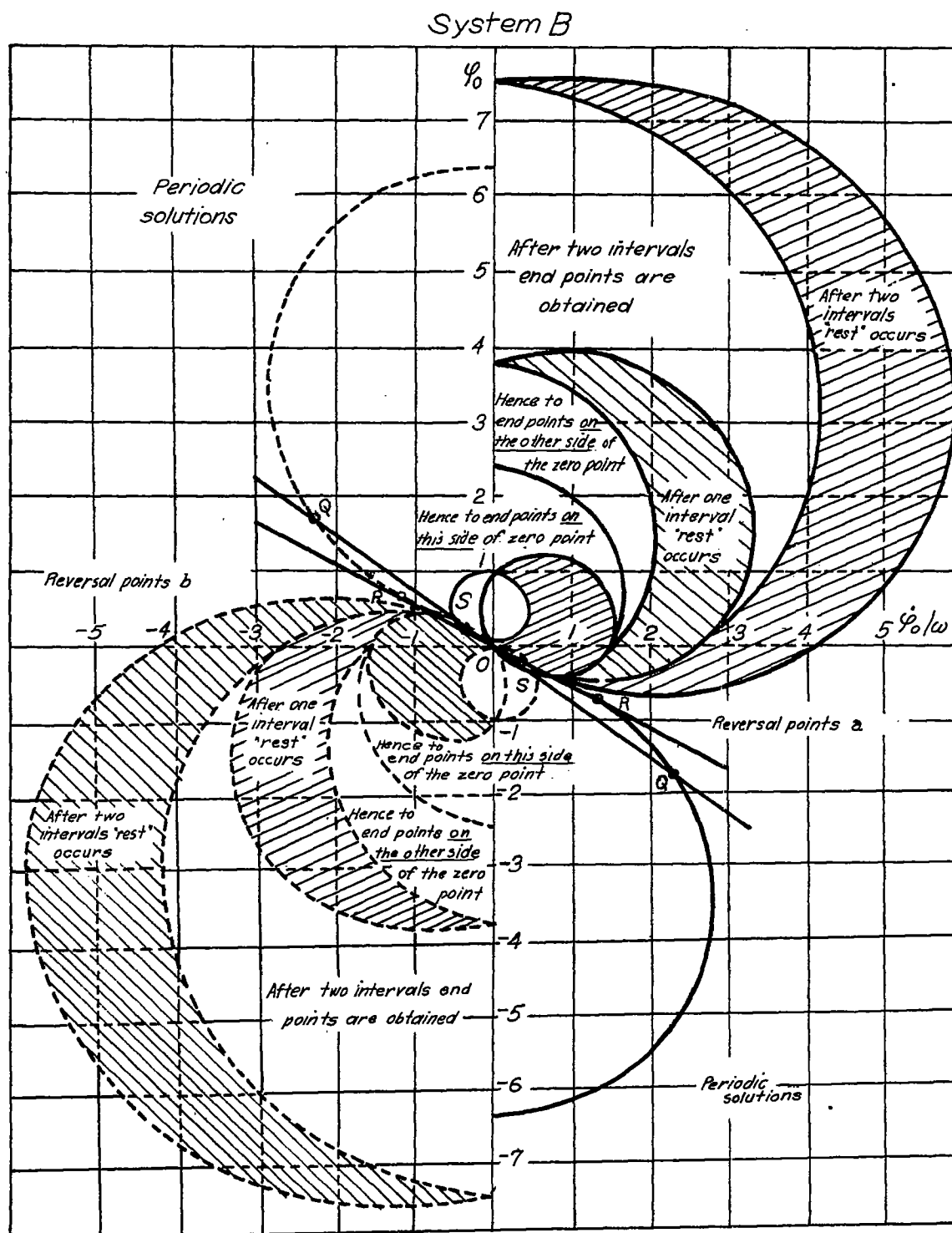


Figure 28(b).- Division of the plane of reversal values. System B.

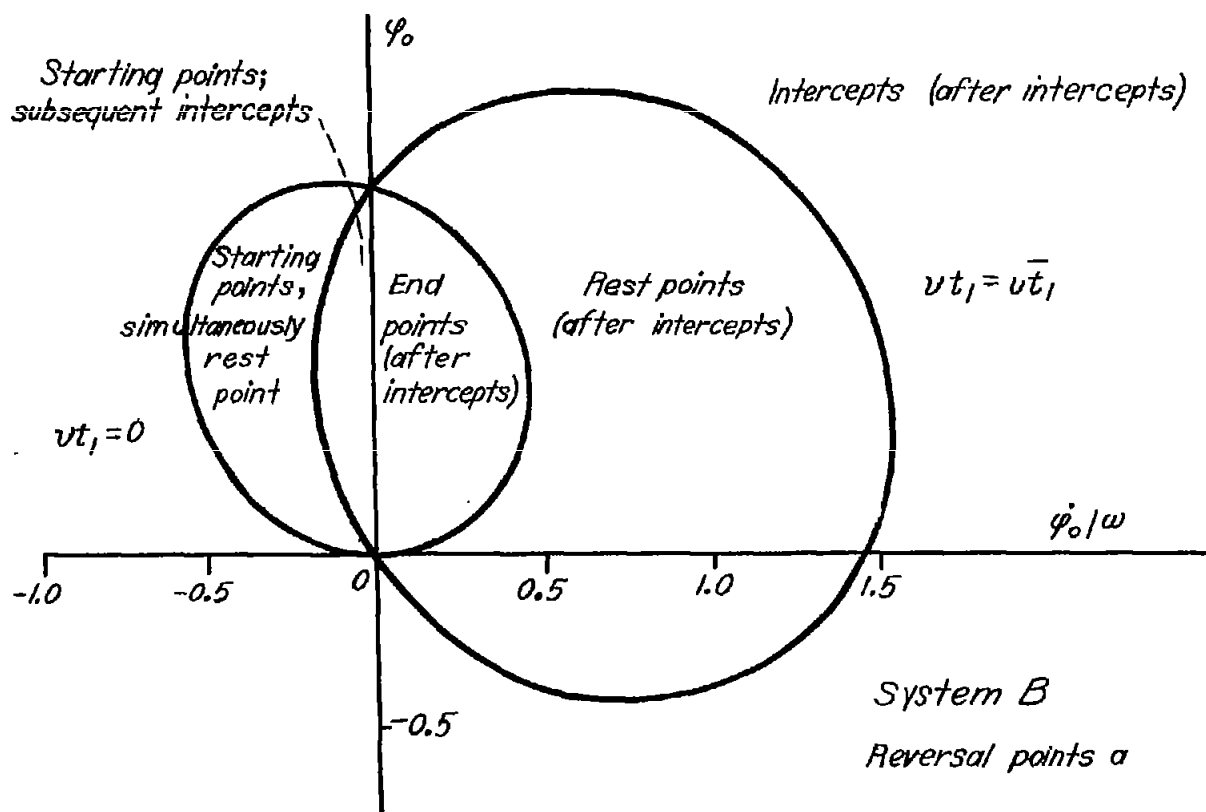


Figure 29.- Large and small limiting ellipses. System B. Reversal points a.